# A Commuting Pair of Truncated Toeplitz Operators & A Local Theory of Stable Polynomials

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This is a story about....

- Rational Inner Functions in 2 variables
- An associated pair of commuting Truncated Toeplitz operators
- How a local theory of stable polynomials is required to understand them.

# Papers

- K. Integrability and regularity of rational functions.
  PLMS (2015)

- Bickel, K., Pascoe, Sola
  Local theory of stable polynomials and bounded rational functions
  arXiv (2021)

# Rational Inner Functions on $\mathbb{D}^2$

$\phi = q/p$

- * $q, p \in \mathbb{C}[z_1, z_2]$
- * $p(z) \neq 0 \quad z \in \mathbb{D}^2$
- * $|q| = |p|$ on $\mathbb{T}^2$

$\phi : \mathbb{D}^2 \longrightarrow \mathbb{D}$

**Thm (Rudin-Stout)** Every RIF is of the form $\phi = \tilde{p}/p$ where

$$\tilde{p}(z) = z_1^{n_1} z_2^{n_2} \overline{p\left(1/\bar{z}_1, 1/\bar{z}_2\right)}$$

and $p, \tilde{p}$ have no common factors. $(n_1, n_2) \geq \text{bideg } p$

Examples:

$$\phi_1 = \frac{3z_1z_2 - z_1 - z_2}{3 - z_1 - z_2}$$

$$\phi_2 = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$$

# Agler Decompositions

$K_1(z;w)$

$\phi = \tilde{p}/p$ RIF on $\mathbb{D}^2$

$\Longrightarrow \exists$ psd polynomial kernels $K_1, K_2$

$$\left.\begin{aligned} &p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} \\ &\quad = (1 - z_1\bar{w}_1)K_1(z;w) + (1 - z_2\bar{w}_2)K_2(z;w) \end{aligned}\right]$$

$$0 \le |p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2)K_1(z,z) + (1 - |z_2|^2)K_2(z,z)$$

$\mathbb{D}^2$

Q: 1. Why are these decompositions important?
2. How can the kernels be constructed?
3. Are they unique?
4. Can all such kernels be described?

## Q ①

Why are these decompositions important?

- "inner-ness" expressed as algebraic positivity certificate.
- Operator substitutions via hereditary functional calculus:

$\rightarrow p(T)^* p(T) - \tilde{p}(T)^* \tilde{p}(T) \geq 0$

$(T_1, T_2) = T$

- Transfer function formulas

$$\phi = \frac{\tilde{p}}{p} = A + B\Delta(1 - D\Delta)^{-1} C$$

$$\Delta = \begin{pmatrix} z_1 I & \\ & z_2 I \end{pmatrix}$$

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ unitary ← finite dimensional

Q: ② How can the kernels be constructed?

- Cone separation argument (existence)
- Kummert's constructive argument 1989
- Hilbert space geometry a la
  - Ball-Sadosky-Vinnikov 2005
  - Geronimo-Woerdeman 2004
  - Bickel-K

See Agler-McCarthy Pick interpolation

Synthesis of two-dimensional lossless m-ports with prescribed scattering matrix, Circuits Systems Signal Process (1989)

Ball, Sadosky, Vinnikov, Scattering systems with several evolutions and multidimensional input/state/output systems. IEOT 2005

Geronimo, Woerdeman Ann. of Math. 2004

Bickel, Knese Inner functions on the bidisk and associated Hilbert spaces. JFA, (2013) Q: (3) Are they unique?

$$|p|^2 - |\tilde{p}|^2 = (1-|z_1|^2)K_1 + (1-|z_2|^2)K_2$$

Thm: if and only if $p$ is <u>saturated</u>:

$$\# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2 = 2n_1n_2$$

Counted using intersection multiplicities

where $\deg \tilde{p} = (n_1, n_2)$.

(Note $\# Z_p \cap Z_{\tilde{p}} \cap \mathbb{C}_\infty^2 = 2n_1n_2$ in general.)

<u>Example:</u> $p = 2 - z_1 - z_2$, $\tilde{p} = 2z_1z_2 - z_1 - z_2$

$Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2 = \{(1,1)\}$ w/ mult. 2. $\longrightarrow$ saturated

$$p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} = \underbrace{(1 - z_1\bar{w}_1) K_1(z;w) + (1 - z_2\bar{w}_2) K_2(z;w)}_{\text{psd kernels}}$$

Agler Decomposition

Q ④ Can all such kernels be described?

I don't know...

but we can describe all minimal dimension kernels via a pair of truncated Toeplitz operators.

$$p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} = (1-z_1\bar{w}_1)K_1(z;w) + (1-z_2\bar{w}_2)K_2(z;w)$$

Agler Decomposition

Let $(n_1, n_2) = \text{bideg}(\tilde{p})$

$(K_1, K_2)$ have minimal dimensions if

$$\text{rank}(K_1) = n_1, \quad \text{rank}(K_2) = n_2$$

(Minimal dimension kernels lead to minimal dimension transfer function realizations)

$1+n_1+n_2$

$$\phi = \tilde{p}/p = A + B\Delta(1-D\Delta)^{-1}C \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

# ACT II

## Hilbert spaces associated to RIF $\phi = \tilde{p}/p$

- $\boxed{\mathcal{H}_\phi} = H^2(\mathbb{T}^2) \ominus \phi H^2(\mathbb{T}^2)$
- $\boxed{K_\phi} = H^2 \cap \phi \overline{(z_1 z_2 H^2)}$ conjugate

$$\overset{*}{=} \left\{ \frac{q}{p} : q \in \mathbb{C}[z_1, z_2],\ \deg q < \deg \tilde{p},\ \frac{q}{p} \in L^2 \right\}$$

$\boxed{\text{Proof} \Leftarrow}$ : $\boxed{p \text{ is outer!!}}$

$q, p \in H^2,\ f = q/p \in L^2 \Rightarrow f \in H^2$

$\deg q < (n_1, n_2) \Rightarrow \tilde{q} := z_1^{n_1 - 1} z_2^{n_2 - 1} \bar{q} \in H^2$

$= \overline{z_1 z_2}\, \tilde{p}\, \bar{f}$

$g := \tilde{q}/p = \frac{\tilde{p}}{p} \overline{z_1 z_2}\, \bar{f} \in L^2 \Rightarrow g \in H^2$

$\Rightarrow f = \overline{z_1 z_2}\, \bar{g}\, \phi$

# Hilbert spaces associated to RIF $\phi = \tilde{p}/p$

THM: $\phi$ has unique Agler decomposition:

$$p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)}$$
$$= (1 - z_1\bar{w}_1) K_1(z;w) + (1 - z_2\bar{w}_2) K_2(z;w)$$

if and only if

$p$ is saturated: $\# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2 = 2n_1n_2$

if and only if

$$K_\phi := H^2 \cap \phi\, \overline{z_1 z_2 H^2} = \{0\}.$$

# Commuting Pair of Truncated Toeplitz Operators

$(T_1^*, T_2)$

- RIF $\phi = \tilde{p}/p$
- $K_\phi = H^2 \cap \phi\overline{(z_1 z_2 H^2)}$

$$T_1 := P_{K_\phi} M_{z_1}, \quad T_2 := P_{K_\phi} M_{z_2}$$

THM: $T_1, T_2^*$ commute!

Proof is somewhat deep... $T_1, T_2$ need not commute!

# Why is the pair $(T_1, T_2^{\circledast})$ interesting?

- Invariant subspaces are in direct correspondence with minimal Agler decompositions

- Joint eigenvalues consist of common zeros of $z_2^{n_2} p(z_1, 1/z_2)$, $z_2^{n_2} \tilde{p}(z_1, 1/z_2)$ within $\underline{\underline{\mathbb{D}^2}}$

Would be interesting to generalize to general inner $\phi$!

- Can use eigenstructure to show

$$\dim K_\phi = n_1 n_2 - \frac{1}{2} \# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$$

$$K_\phi = \{ q : \deg q < (n_1, n_2),\ q/p \in L^2 \}$$

# Easiest Non-trivial Example

$p = 1$, $\tilde{p} = z_1^n z_2^n$, $\boxed{\phi = z_1^n z_2^n}$

$$K_\phi = \text{span}\{ z_1^j z_2^k : 0 \le j, k < n \} \subseteq H^2$$

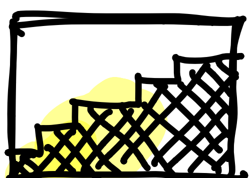
$$T_1 = P_{K_\phi} M_{z_1} : z_1^j z_2^k \longmapsto \begin{cases} z_1^{j+1} z_2^k & j+1 < n \\ 0 & j = n-1 \end{cases}$$

$$T_2^* = P_{K_\phi} M_{z_2}^* : z_1^j z_2^k \longmapsto \begin{cases} z_1^j z_2^{k-1} & k > 1 \\ 0 & k = 0 \end{cases}$$

1 joint eigenvalue $(0,0)$

Joint generalized eigenspace $= K_\phi$

What are the invariant subspaces?

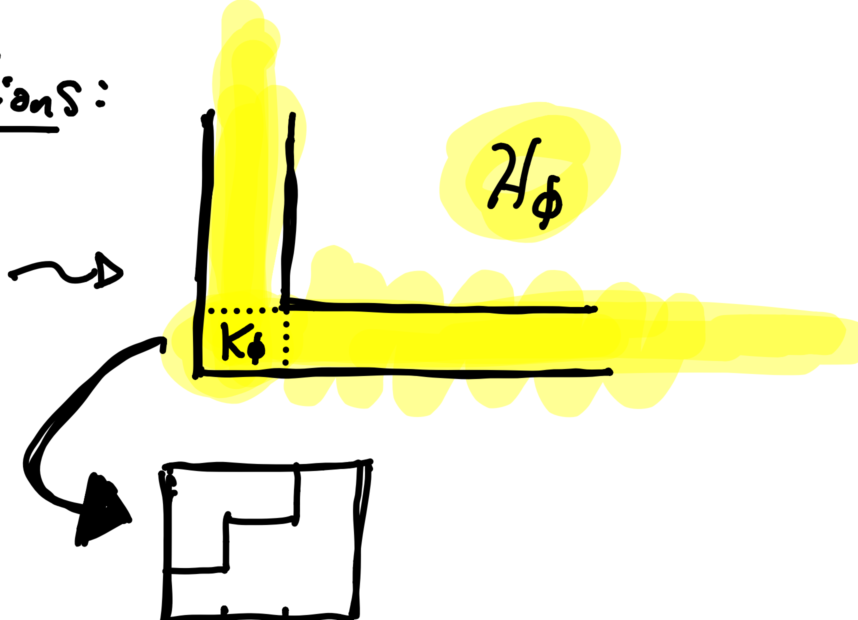
$\{0\}$, $K_\phi$,  $\longleftarrow$ All staircases from $(0,0)$ to $(n,n)$

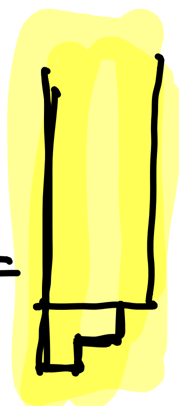
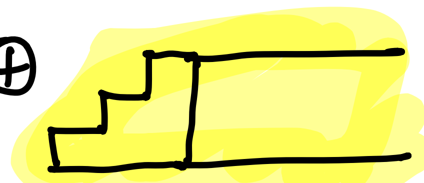
# Easier Non-trivial Example

$p = 1$, $\tilde{p} = z_1^3 z_2^3$, $\phi = z_1^3 z_2^3$

## Agler Decompositions:

$$\frac{1 - |z_1^3 z_2^3|^2}{(1-|z_1|^2)(1-|z_2|^2)}$$




$\mathcal{H}_\phi =$  $\oplus$ 

$$= \bigoplus_{j=0}^{\infty} z_2^j \vee \{z_2, z_1 z_2^2, z_2 z_2^3\}$$

$$\oplus \bigoplus z_1^j \vee \{1, z_1 z_2, z_1^2 z_2^2\}$$

$$\frac{1-|z_1^3 z_2^3|^2}{(1-|z_1|^2)(1-|z_2|^2)} = \frac{|z_2|^2\left(1+|z_1 z_2|^2+|z_1^2 z_2^2|^2\right)}{1-|z_2|^2}$$

$$+\frac{1+|z_1 z_2|^2+|z_1^2 z_2^2|^2}{1-|z_1|^2}$$

$$\Rightarrow \quad 1-|z_1^3 z_2^3|^2 = (1-|z_1|^2)\left(|z_2|^2+|z_1 z_2^2|^2+|z_1^2 z_2^3|^2\right)$$

$$+(1-|z_2|^2)\left(1+|z_1 z_2|^2+|z_1^2 z_2^2|^2\right)$$

# Act III: Local theory of stable polynomials

① $\# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$ is even.

Why?

② $\mathcal{I}_p^2 = \{ q \in \mathbb{C}[z_1, z_2] : q/p \in L^2 \}$

$= ?$

③ When is $q/p \in L^\infty$?

# Reduction to local questions

RIF $\phi = \tilde{p}/p$

- WLOG $\gcd(p, \tilde{p}) = 1$

  $\Rightarrow Z_p \cap \mathbb{T}^2 = Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$ finite

- $p$ called <u>atoral stable</u>
- Zeros on $\mathbb{T}^2$ isolated so examine single zero.
- Perform Cayley transform and examine

  $p \in \mathbb{C}[z_1, z_2]$, $p(z) \neq 0$ $\quad z \in \mathbb{H}^2$ upper halfplane

  $\gcd(p, \bar{p}) = 1$

  $p(0,0) = 0$

  "pure stable"

# Pure stable Puiseux factorization (BKPS)

$p \in \mathbb{C}[z_1, z_2]$ pure stable, $p(0,0) = 0$.

convergent power series

Then, $p = u\, p_1 \cdots p_k$, $u \in \mathbb{C}\{z_1, z_2\}$, $u(0,0) = 0$

$p_j \in \mathbb{C}[z_2]\{z_1\}$

$$p_j = \prod_{m=1}^{M_j} \left( z_2 + q_j(z_1) + z_1^{2L_j}\, \psi_j\!\left(\mu_j^m z_1^{1/M_j}\right)\right)$$

**DATA**:

$L_j, M_j \in \mathbb{N}$, $\mu_j = e^{2\pi i/M_j}$

$q_j \in \mathbb{R}[z_1]$, $\deg q_j < 2L_j$, $q_j(0) = 0$, $q_j'(0) > 0$

$\psi_j \in \mathbb{C}\{t\}$, $\operatorname{Im} \psi_j(0) > 0$

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**COR**: $[p] := \prod_{j=1}^{k} \left( z_2 + q_j(z_1) + i\, z_1^{2L_j}\right)^{M_j}$

$\Longrightarrow$ $p/[p]$, $[p]/p$ bounded near $(0,0)$ in $\mathbb{R}^2$.

① $\# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$ is even.

Why?

Multiplicity of common zero of $p$ and $\bar{p}$ at $(0,0)$ computed via

$$\operatorname{ord} \prod_{j,k} \left( \alpha_j(z_1) - \bar{\alpha}_k(z_1) \right)$$

where $z_2 \mapsto p(z_1, z_2)$ has roots $\alpha_1(z_1), \dots, \alpha_M(z_1)$

$z_2 \mapsto \bar{p}(z_1, z_2)$ $\qquad \bar{\alpha}_1(z_1), \dots, \bar{\alpha}_M(z_1)$

Using Puiseux, multiplicity associated to single place:

$$\operatorname{ord} \prod_{n,m} \begin{pmatrix} -\left(\cancel{\xi_j(z_1)} + z_1^{2L_j} \Psi_j\left(\mu_j^m z_1^{1/M_j}\right)\right) \\ + \left(\cancel{\xi_j(z_1)} + z_1^{2L_j} \overline{\Psi_j}\left(\mu_j^n z_1^{1/M_j}\right)\right) \end{pmatrix} = 2 L_j M_j^2$$

② $\mathcal{I}_p^2 = \left\{ q \in \mathbb{C}[z_1, z_2] : q/p \in L^2(\mathbb{T}^2) \right\} = ?$

- Can construct generators from Agler kernels
- Local approach can also be followed

$\hookrightarrow$ Switch to $p(z) \neq 0$ $z \in \mathbb{H}^2$ pure stable

Sample result: If $p$ vanishes to order 1 then $\{ q \in \mathbb{C}[z_1, z_2] : q/p \in L^2_{loc}(0,0) \}$

$$= \{ x^L f_0(x) : \deg f_0 < L \} + (p, \bar{p})$$

$L$ from Puiseux data

## ③ When is $q/p \in L^\infty(\mathbb{T}^2)$?

- Of interest for constructing non-inner bounded rational functions
- Again, can reduce to local question in $\mathbb{H}^2$ setting: $p(z) \neq 0$ $z \in \mathbb{H}^2$, pure stable.

Sample result: If $p$ vanishes to order 1 at $(0,0)$, then

$$\{q \in \mathbb{C}[z_1, z_2] : q/p \in L^\infty_{loc}(0,0)\}$$

$$= (p, \bar{p})$$

# Papers

- K. Integrability and regularity of rational functions.
  PLMS (2015)

- Bickel, K., Pascoe, Sola
  Local theory of stable polynomials and bounded rational functions
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# The End