AN OPERATOR-VALUED VERSION OF V.P. POTAPOV'S MATRIX-VALUED FACTORIZATION RESULT

(WITH IN SUNG HWANG AND WOO YOUNG LEE)

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ABSTRACT

We consider six questions emerging from the Beurling-Lax-Halmos Theorem, which characterizes the shift-invariant subspaces of vector-valued Hardy spaces:

A backward shift-invariant subspace is a model space $\mathcal{H}(\Delta) \equiv H^2_E \ominus \Delta H^2_E$, for some inner function Δ.

Question 1: For a set $F \subseteq H^2_E$, let E^*_F denote the smallest backward shift-invariant subspace containing F. Thus, there exists Δ inner such that $\mathcal{H}(\Delta) = E^*_F$. What is the smallest number of vectors in F satisfying this equation? More generally, describe F such that $\mathcal{H}(\Delta) = E^*_F$.

To examine Question 1, we consider operator-valued functions on the unit circle $\mathbb{T}$ constructed by arranging the vectors of F as column vectors.

In our pursuit of a general solution to this question, we are naturally led to take into account a new canonical decomposition of operator-valued strong L^2-functions.

Question 2: Is every strong L^2–function Φ of the form $\Phi = \Delta A^*$, for some inner function Δ ?

Our description includes, as a special case, the Douglas-Shapiro-Shields factorization for matrix functions of bounded type.

Question 3: Is every shift-invariant subspace the kernel of a (possibly unbounded) Hankel operator ?

Question 3 leads naturally to a new notion of “Beurling degree" for an inner function.

Question 4: How is the Beurling degree of Δ related to the spectral multiplicity of $S_E^*|_{\mathcal{H}(\Delta)}$?

Next, we will consider meromorphic continuations of bounded type for operator-valued functions, and use this notion to study the spectral multiplicity of model operators, and ask:

Question 5: Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. For which inner function Δ does it follow that T is multiplicity-free?

Finally, we will discuss an operator-valued extension of V.P. Potapov's celebrated theorem:

An $n \times n$ matrix function is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

Question 6: Is Potapov's Theorem still true for operator-valued functions?

We will prove that if Δ is a left-inner divisor of the coordinate function zI_E, then Δ is a Blaschke-Potapov factor. This requires a new notion of operator-valued rational function in the infinite multiplicity case.

Overview

$\mathbb{T}$: unit circle

$\mathbb{D}$: open unit disk

$L^2 \equiv L^2(\mathbb{T})$, $H^2 \equiv H^2(\mathbb{T})$, $L^\infty \equiv L^\infty(\mathbb{T})$, $H^\infty := L^\infty \bigcap H^2$.

P, $P^\perp$: orthogonal projections from L^2 to H^2 and $(H^2)^\perp$, resp.

Given $\varphi \in L^\infty$, the Toeplitz (resp. Hankel) operator acting on H^2 is defined by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2),$$

(resp.

$$H_\varphi f := JP^\perp(\varphi f)) \quad (f \in H^2),$$

where J is the unitary operator on L^2 defined by $(Jf)(z) := \bar{z} f(\bar{z})$.

T_φ is said to be *analytic* if $\varphi \in H^\infty$.

Functions of bounded type

$\varphi \in L^\infty$ is of *bounded type* (or in the Nevanlinna class $\mathcal{N}$) if

$$\varphi := \frac{\psi_1}{\psi_2} \quad (\psi_1, \psi_2 \in H^\infty).$$

Halmos's Problem 5 (1970):

Is every subnormal Toeplitz operator either normal or analytic?

(C. Cowen and J. Long, 1984): No, and they gave a concrete example.

(Abrahamse, 1976) Assume φ or $\bar{\varphi}$ is of bounded type. If T_φ is hyponormal and $\ker[T_\varphi^*, T_\varphi]$ is invariant for T_φ, then T_φ is normal or analytic.

Thus, the answer to Halmos's Problem 5 is affirmative if φ is of bounded type.

Beurling's Theorem states that a nontrivial shift-invariant subspace $\mathcal{M} \subseteq H^2$ must be of the form

$$\mathcal{M} = \theta H^2,$$

where θ is an inner function. If $0 \neq f \in H^2$ and we form the smallest invariant subspace $\mathcal{M}_f$ containing f, Beurling's Theorem implies that there exists an inner function θ such that $\mathcal{M}_f = \theta H^2$, and therefore

$$f = \theta g,$$

for some $g \in H^2$. In fact, g is a cyclic vector for the shift $S \equiv T_z$. This produces the **inner-outer factorization** of f, since outer functions are the cyclic vectors of S.

Beurling also proved that, if $g \in H^2$, then

$$g \text{ is outer } \Longleftrightarrow \log\ |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\ \left|g(e^{it})\right|\ dt\ .$$

Given a bounded operator T on Hilbert space, a closed subspace $\mathcal{M}$ is invariant under T^* if and only if $\mathcal{M}^\perp$ is invariant under T. Thus, associated with an inner function θ (which produces a shift-invariant subspace θH^2) is the model space $\mathcal{H}(\theta) := H^2 \ominus \theta H^2$ (which is backward shift-invariant).

The above observation falls short of determining which $f \in H^2$ can be cyclic vectors for S^*. For this, something deeper is needed.

(R.G. Douglas, J. Shapiro and A. Shields, 1970) $f \in H^2$ is non-cyclic for S^* if and only if there exists $g \in H^2$ and an inner function θ such that

$$f(z) = \bar{z}\theta(z)\overline{g(z)} \text{ (a.e. on } \mathbb{T}).$$

This is equivalent to requiring that f has a meromorphic pseudo-continuation of bounded type to the exterior of the unit disk. Moreover, if θ and g are coprime, we obtain a certificate of non-cyclicity: $(\bigvee_{n \geq 1} S^{*n} f)^\perp = \theta H^2$.

(Recall: θ and g are coprime if they do not have a common nontrivial factor.)

$M_n := M_{n\times n}$ $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$ $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ For $\Phi \in L^\infty_{M_n}$, $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the *block* Toeplitz operator with symbol Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$.

A *block Hankel* operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$.

For example, if $f \equiv f_0 + zf_1 + z^2 f_2 + z^3 f_3 + \ldots \in H^2_{\mathbb{C}^n}$ $(f_i \in \mathbb{C}^n)$, then

$$H_{\overline{z}^3} f = f_2 + zf_1 + z^2 f_0.$$

We easily see that

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \dots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \dots & T_{\varphi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \dots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \dots & H_{\varphi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \dots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := (\Phi(\overline{z}))^*.$$

A matrix-valued function $\Delta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Delta^*\Delta = I_m$ almost everywhere on $\mathbb{T}$. Given $\Phi, \Psi \in L^\infty_{M_n}$,

$$T^*_\Phi = T_{\Phi^*}, \quad H^*_\Phi = H_{\widetilde{\Phi}} \quad \text{(recall that } \widetilde{\Phi}(z) := (\Phi(\overline{z}))^*)$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi$$

$$H_\Phi T_\Psi = H_{\Phi\Psi} \quad (\Psi \in H^\infty)$$

$$H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Psi \in H^\infty).$$

As a result,

$$H_{\Phi^*} T_z = T_z^* H_{\Phi^*},$$

and $\ker H_{\Phi^*}$ is an invariant subspace of the shift operator T_z .

$\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$ is of *bounded type* if each entry φ_{ij} is of bounded type.

Φ is *rational* if each entry φ_{ij} is a rational function.

FOR Δ INNER, Δ^* MAY NOT BE OF BOUNDED TYPE

Let

$$h(z) := e^{\frac{1}{z-3}}$$

and

$$f(z) := \frac{h(z)}{\sqrt{2}\,\|h\|_\infty}.$$

$h \in H^\infty$ and $\bar{h}$ is not of bounded type, and therefore $\bar{f}$ is not of bounded type. Now consider

$$h_1(z) := \sqrt{1 - |f(z)|^2}.$$

Then $h_1 \in L^\infty$ and

$$|h_1| \geq \frac{1}{\sqrt{2}}.$$

Thus, there exists an outer function g such that $|h_1| = |g|$ a.e. on $\mathbb{T}$.

Let

$$\Delta := \begin{bmatrix} f \\ g \end{bmatrix} \quad (f, g \in H^\infty).$$

Then

$$\Delta^*\Delta = |f|^2 + |g|^2 = |f|^2 + |h_1|^2 = 1 \text{ a.e. on } \mathbb{T},$$

which implies that Δ is an inner function. However, Δ^* is not of bounded type.

Block Toeplitz operators have been studied by D.Z. Arov, J. Ball, E. Basor, W. Bhosri, V. Bolotnikov, A. Böttcher, R.G. Douglas, H. Dym, I. Feldman, A. Frazho, P. Fuhrmann, I. Gohberg, S. Grudsky, C. Gu, A. Hartmann, W. Helton, J. Hendricks, I.S. Hwang, D.-O. Kang, M.A. Kaashoek, I. Koltracht, W.Y. Lee, N.K. Nikolskii, V. Peller, G. Popescu, A. Rogozhin, D. Rutherford, H. Shapiro, A. Shields, I. Spitkovsky, S. ter Horst, V. Vinnikov, H. Woerdeman, D. Yakubovich, D. Zheng, K. Zhu, Y. Zucker, and many others. R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, Amer. Math. Soc., 1980.

The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_z .$$

The Beurling-Lax-Halmos Theorem (BLH Theorem). *A nonzero subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^n}$ is invariant for S if and only if $\mathcal{M} = \Theta H^2_{\mathbb{C}^m}$ $(m \leq n)$, where Θ is an inner matrix function. Furthermore, Θ is unique up to a unitary constant right factor.*

As a consequence, if $\ker H_\Phi \neq \{0\}$, then

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ.

The BLH Theorem is true if $\mathbb{C}^n$ is replaced by a separable infinite dimensional Hilbert space, as shown by P.R. Halmos.

Theorem

(Gu, Hendricks and Rutherford, 2006) *For* $\Phi \in L^\infty_{M_n}$*, the following statements are equivalent:*

1. Φ *is of bounded type;*
2. $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ *for some square inner matrix function* Θ;
3. $\Phi = A\Theta^*$*, where* $A \in H^\infty_{M_n}$ *and* A *and* Θ *are right coprime.*

Definition: Θ and A are right coprime if they do not have a common nontrivial right factor.

Is Abrahamse's Theorem valid for Toeplitz operators with matrix-valued symbols?

In general, a straightforward matrix-valued version of Abrahamse's Theorem is doomed to fail: for instance, if

$$\Phi := \begin{bmatrix} z+\overline{z} & 0 \\ 0 & z \end{bmatrix},$$

then both Φ and Φ^* are of bounded type and

$$T_\Phi = \begin{bmatrix} T_z + T_z^* & 0 \\ 0 & T_z \end{bmatrix}$$

is subnormal, but neither normal nor analytic.

In 2014, with Dong-O Kang (CHKL), we proved a matrix-valued version of Abrahamse's theorem, in the *rational* symbol case. Later on, we extended this result to the case of *bounded type* symbols, and obtained a full-fledged matrix-valued version of Abrahamse's Theorem.

DEFINITION

A symbol Φ has a matrix singularity if $\ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^n}$ for some nonconstant inner function θ.

THEOREM

(Abrahamse's Thm. for matrix-valued symbols) Let $\Phi \in L^\infty_{M_n}$ with Φ and Φ^* of bounded type. Assume Φ has a matrix singularity. If

(I) *T_Φ is hyponormal;*

(II) *$\ker\,[T^*_\Phi, T_\Phi]$ is invariant under T_Φ,*

then T_Φ is normal. In particular, if T_Φ is subnormal then T_Φ is normal or analytic.

D, E: separable complex Hilbert spaces
L^2_E: E–valued L^2 space (alternatively, $L^2 \otimes E$)
H^2_E: E–valued Hardy space
Φ: operator-valued function on $\mathbb{T}$, mapping $z \in \mathbb{T}$ to $\Phi(z) \in \mathcal{B}(D, E)$.

A *strong L^2–function* Φ is a $\mathcal{B}(D, E)$–valued function defined almost everywhere on $\mathbb{T}$ such that $\Phi(\cdot)x \in L^2_E$ for each $x \in D$.

Strong L^2–functions have been considered by N. Nikolskii and V. Peller. In particular, Peller shows that the set $L^2_s(\mathcal{B}(D, E))$ of strong L^2_E–functions constitutes a nice collection of symbols of vectorial Hankel operators.

The set $H^2_s(\mathcal{B}(D, E))$ has the obvious definition.

Lemma

Let Φ be a strong L^2-function with values in $\mathcal{B}(D, E)$. Then

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}, \tag{1}$$

where $\breve{\Phi}(z) := \Phi(\bar{z})$ is the flip of Φ, E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E', E)$.
(The Hankel operator $H^*_{\breve{\Phi}}$ may be unbounded.)

For an inner function $\Delta \in H^\infty(\mathcal{B}(E', E))$, $\mathcal{H}(\Delta)$ denotes the orthogonal complement of the subspace $\Delta H^2_{E'}$ in H^2_E, i.e.,

$$\mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_{E'}.$$

The space $\mathcal{H}(\Delta)$ is often called a *model space* or a *de Branges-Rovnyak space*.

Observation. If Φ is an operator-valued L^∞–function, then the kernel of the Hankel operator with symbol Φ^* is shift-invariant. By the BLH Theorem, it must be of the form $\Delta H^2_{E'}$ for some inner function Δ. Now, Δ is not necessarily a two-sided inner function. In fact, if it is, then

$$\Phi = \Delta A^*, \tag{2}$$

where A is an operator-valued H^∞ function and Δ and A are right coprime. The above factorization is the (canonical) Douglas-Shapiro-Shields factorization.

Later on, we'll see a version of this result for operator-valued H^∞–function.

LEMMA

Equation (2) above characterizes the class of operator-valued L^∞–functions Φ whose flips $\breve{\Phi}$ are of bounded type, where $\breve{\Phi}(z) := \Phi(\bar{z})$.

With the aid of this result, we can prove:

An answer to Question 2.

THEOREM

(Canonical Decomposition of Strong L^2–functions) Let Φ be a strong L^2–function with values in $\mathcal{B}(D,E)$. Then

$$\Phi = \Delta A^* + B, \qquad (*)$$

where

(I) Δ *is an inner function with values in* $\mathcal{B}(E',E)$, $E' \subseteq E$,

(II) Δ *and* A *are right coprime,*

(III) $\Delta^* B = 0$, *and*

(IV) $nc\{\Phi_+\} \leq \dim E'$, *where* $\{\Phi_+\}$ *is the set of column vectors of the analytic part of* Φ *and* nc *is the* **degree of non-cyclicity** *(introduced by V.I. Vasyunin and N. Nikolskii).*

If $\dim E' < \infty$, *then* $(*)$ *is unique (up to a unitary constant right factor).*

If $\Phi \in H^2_s(\mathcal{B}(D,E))$ and $\{d_k\}_{k\geq 1}$ is an orthonormal basis for D, write

$$\phi_k := \Phi d_k \in H^2_E \cong H^2_s(\mathcal{B}(\mathbb{C},E)).$$

We then define

$$\{\Phi\} := \{\phi_k\}_{k\geq 1} \subseteq H^2_E.$$

Hence, $\{\Phi\}$ may be regarded as the set of “column" vectors ϕ_k (in H^2_E), in which case we may think of Φ as an infinite matrix-valued function.

We can visualize this as

$$\Phi = \begin{bmatrix} (\phi_1)_0 & (\phi_2)_0 & \dots & (\phi_k)_0 & \dots \\ (\phi_1)_1 & (\phi_2)_1 & \dots & (\phi_k)_1 & \dots \\ (\phi_1)_2 & (\phi_2)_2 & \dots & (\phi_k)_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If, in addition, we know that $\Phi = \Delta A^* + B$ (for Δ inner and $B \in \ker \Delta^*$), then $\Delta^*\Phi = A^*$, and therefore $\Phi = \Delta\Delta^*\Phi + B$ $(\text{ with } B \in \ker \Delta^*)$.

Recall

LEMMA

Let Φ be a strong L^2-function with values in $\mathcal{B}(D, E)$. Then

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}, \qquad (3)$$

where $\breve{\Phi}(z) := \Phi(\bar{z})$ is the flip of Φ, E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E', E)$.
(The Hankel operator $H^*_{\breve{\Phi}}$ may be unbounded.)

Definition

We will say that the inner function Δ_c is the *complementary factor* of the inner function Δ if

$$[\Delta \;\; \Delta_c] \begin{bmatrix} \Delta^* \\ \Delta_c^* \end{bmatrix} = I.$$

Corollary

If Φ is an $n \times m$ matrix L^2-function, i.e., $\Phi \in L^2_{M_{n\times m}}$, then the following are equivalent:

(A) *Φ is of bounded type;*

(B) $\ker H^*_\Phi = \Delta H^2_{\mathbb{C}^n}$ *for some two-sided inner matrix function Δ.*

Corollary

If Δ is an $n \times r$ inner matrix function then the following are equivalent:

(A) *Δ^* is of bounded type;*

(B) *$\breve{\Delta}$ is of bounded type;*

(C) *$[\Delta \;\; \Delta_c]$ is two-sided inner, i.e., $\Delta^* \Delta_c = 0$.*

Recall

Question 1: For a set $F \subseteq H^2_E$, let E^*_F denote the smallest backward shift-invariant subspace containing F. Thus, there exists Δ such that $\mathcal{H}(\Delta) = E^*_F$. What is the smallest number of vectors in F satisfying this equation? More generally, describe F such that $\mathcal{H}(\Delta) = E^*_F$.

Theorem

(An answer to Question 1) Let $\Phi \in L^2_s(\mathcal{B}(D, E))$, and denote by $\{\Phi_+\}$ the set of column vectors of the analytic part of Φ. The following are equivalent:

(i) $\breve{\Phi}$ is of bounded type;

(ii) $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$ for some two-sided inner function Δ with values in $\mathcal{B}(E)$;

(iii) $\{\Phi_+\} \subseteq \mathcal{H}(\Theta)$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$;

(iv) For $\{\varphi_{k_1}, \varphi_{k_2}, \dots\} \subseteq \{\Phi\}$, let $\Psi := [\varphi_{k_1}, \varphi_{k_2}, \dots]$. Then $\breve{\Psi}$ is of bded. type.

The following lemma gives a characterization of bounded Hankel operators on H^2_D.

Lemma

(cf. V. Peller's book, 2003) Let $\Phi \in L^2_s(B(D,E))$. Then H_Φ can be extended to a bounded operator on H^2_D if and only if there exists a function $\Psi \in L^\infty(\mathcal{B}(D,E))$ such that $\widehat{\Psi}(n) = \widehat{\Phi}(n)$ for $n < 0$ and

$$||H_\Phi|| = dist_{L^\infty}(\Psi, H^\infty(\mathcal{B}(D,E)).$$

Recall

Question 3: Is every shift-invariant subspace the kernel of a (possibly unbounded) Hankel operator ?

Theorem

(An answer to Question 3) Let Δ be an inner function with values in $\mathcal{B}(E', E)$. Then there exists a function Φ in $H^2_s(\mathcal{B}(D,E))$, with either $D = E'$ or $D = \mathbb{C} \oplus E'$, satisfying $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$.

MEROMORPHIC PSEUDO-CONTINUATIONS OF BOUNDED TYPE

A $\mathcal{B}(D,E)$-valued function Ψ is said to be *meromorphic of bounded type* in $\mathbb{D}^e$ if it can be represented by

$$\Psi = \frac{G}{\theta},$$

where G is a strong H^2-function in $\mathbb{D}^e$, with values in $\mathcal{B}(D,E)$ and θ is a scalar inner function in $\mathbb{D}^e$. A function $\Phi \in L^2_s(\mathcal{B}(D,E))$ is said to have a *meromorphic pseudo-continuation* $\hat{\Phi}$ *of bounded type* in $\mathbb{D}^e$ if $\hat{\Phi}$ is meromorphic of bounded type in $\mathbb{D}^e$ and Φ is the nontangential SOT limit of $\hat{\Phi}$, that is, for all $x \in D$,

$$\Phi(z)x = \hat{\Phi}(z)x := \lim_{rz \to z} \hat{\Phi}(rz)x \quad \text{for almost all } z \in \mathbb{T}.$$

PROPOSITION

Let D and E be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of D and E, respectively. If $\Phi \in L^2_{\mathcal{B}(D,E)}$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then $\breve{\phi}_{ij}(z) \equiv \langle \breve{\Phi}(z) d_j, \ e_i \rangle_E$ is of bounded type for each i, j.

COROLLARY

For $\Phi \equiv [\phi_{ij}] \in L^2_{M_{n\times m}}$, the following are equivalent:

(A) Φ *has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$;*

(B) $\breve{\Phi}$ *is of bounded type;*

(C) $\breve{\phi}_{ij}$ *is of bounded type for each i, j.*

DEFINITION

Let Δ be an inner function with values in $\mathcal{B}(E', E)$. Then the *Beurling degree* of Δ, denoted by $\deg_B(\Delta)$, is defined by

$$\deg_B(\Delta) := \inf\Big\{\dim D \in \mathbb{Z}_+ \cup \{\infty\} : \text{ there exists a pair } (A, B) \quad \text{s.t. } \Phi = \Delta A^* + B \text{ is a canonical decomposition of } \Phi \in L^2_s(B(D, E))\Big\}$$

Definition

The *spectral multiplicity* for a bounded linear operator T acting on a separable complex Hilbert space E is defined as

$$\mu_T := \inf \dim F,$$

where $F \subseteq E$, the infimum being taken over all generating subspaces F, i.e., subspaces such that $M_F \equiv \bigvee \{T^n F : n \geq 0\} = E$.

Answers to Question 4 and Question 5.

THEOREM

(The Beurling degree and the spectral multiplicity) Given an inner function Δ with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$, let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Then

$$\mu_T = \deg_B(\Delta). \tag{4}$$

COROLLARY

Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. If rank $(I - T^*T) < \infty$, then

$$\mu_T = \deg_B(\Delta).$$

For $\alpha \in \mathbb{D}$, write

$$b_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z},$$

which is called a *Blaschke factor*. If M is a closed subspace of a Hilbert space E, then a function of the form

$$b_\alpha P_M + (I_E - P_M)$$

is called a *(operator-valued) Blaschke-Potapov factor*, where P_M is the orthogonal projection of E onto M.

A function D is called a *(operator-valued) finite Blaschke-Potapov product* if D is of the form

$$D = V \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m) \Big),$$

where V is a unitary operator, b_m is a Blaschke factor, and P_m is an orthogonal projection in E for each $m = 1, \cdots, M$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{M} b_m$, where $\nu = e^{i\omega}$.

(V.P. Potapov, 1955) An $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

Natural Question: What is a left inner divisor of $z \cdot I_n$?

We may initially guess that each left inner divisor of $z \cdot I_n$ is a Blaschke-Potapov factor. More specifically, we wonder if a left inner divisor of $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] \equiv z \cdot I_2$ should be of the following form up to a unitary constant right factor (also up to unitary equivalence):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.$$

For example, $A \equiv \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -z \\ 1 & z \end{smallmatrix}\right]$ is a left inner divisor of $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] \equiv z \cdot I_2$: indeed,

$$A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & z \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.$$

In this case, if we take a unitary operator $V := \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} = \left[V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} \cdot V^* \right] \cdot V.$$

In 2012, we proved that

every left inner divisor of $z \cdot I_n \in H^\infty(\mathbb{T}, M_n)$ is a Blaschke-Potapov factor. (5)

This fact is useful for the study of coprime-ness of functions.

Question

Is the statement in (5) still true for operator-valued functions?

If $\dim E < \infty$ and $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is a two-sided inner function, then any left inner divisor of Θ is two-sided inner (RC, Hwang & Lee, MAMS, 2019). We can say more:

LEMMA

(RC, Hwang & Lee, JFA, 2021) Let $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a two-sided inner function. If $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is a left inner divisor of Θ, then Δ is two-sided inner.

In particular, if $\Theta := \theta \cdot I_E$ (for θ a scalar inner function) then every left inner divisor of Θ is an inner divisor of Θ. However, in general, a left inner divisor of a two-sided inner function need not be its right inner divisor. To see this, we first observe:

LEMMA

Let $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a two-sided inner function and $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a left inner divisor of Θ. Then Δ is an inner divisor of Θ if and only if $\Theta\Delta^* \in H^\infty(\mathbb{T}, \mathcal{B}(E))$.

Example

Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $L^2(\mathbb{T})$. Define Δ and Θ in $H^\infty(\mathbb{T}, \mathcal{B}(L^2(\mathbb{T})))$ by

$$\Delta(z)e_n := \begin{cases} e_{n+1}z & \text{if } n \geq 0 \\ e_{n+1} & \text{if } n < 0 \end{cases} \quad \text{and} \quad \Theta(z)e_n := \begin{cases} e_{-n+1}z^2 & \text{if } n \leq 1 \\ e_{-n+1} & \text{if } n > 1. \end{cases}$$

Then Θ and Δ are two-sided inner. Observe that

$$\Delta^*(z)e_n = \begin{cases} e_{n-1}z^{-1} & \text{if } n \geq 1 \\ e_{n-1} & \text{if } n < 1 \end{cases} \quad \text{and hence} \quad \Delta^*(z)\Theta(z)e_n = \begin{cases} e_{-n}z & \text{if } n < 1 \\ e_{-1}z^2 & \text{if } n = 1 \\ e_{-n} & \text{if } n > 1. \end{cases}$$

Thus Δ is a left inner divisor of Θ. However, since

$$\Theta(z)\Delta^*(z)e_3 = e_{-1}z^{-1},$$

it follows that Δ is not a right inner divisor of Θ.

EXAMPLE

Let S be the shift operator on $H^2(\mathbb{T})$ defined by

$$(Sf)(z) := z \cdot f(z) \quad (f \in H^2(\mathbb{T}),\ z \in \mathbb{T})$$

and let $\Delta(z) := S \in H^\infty(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$. Then

$$\Delta(z)^* \Delta(z) = S^* S = I,$$

which implies that Δ is a right inner divisor of (a two-sided inner function) I. But Δ is not two-sided inner. Therefore, Δ is not a left inner divisor of I.

Definition

A function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$ is said to be *rational* if

$$\theta H^2(\mathbb{T}, E) \subseteq \ker H_{\Phi^*} \tag{6}$$

for some finite Blaschke product θ.

Theorem

Let θ be a finite Blaschke product. If Δ is an inner divisor of $\Theta = \theta I_E$, then Δ is a finite Blaschke-Potapov product.

Recall: If $\Phi \in L^\infty_E$ then $\ker H_{\Phi^*} = \Delta H^2_E$ for some Δ inner.

(Douglas-Shapiro-Shields) If Δ is two-sided inner, then $\Phi = \Delta A^*$, where $A \in H^\infty_E$ and A, Δ are right coprime.

We now consider the case of $\Phi \in H^\infty_E$.

Corollary

A function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$ is rational if and only if

$$\Phi = \Delta A^*,$$

where Δ is a finite Blaschke-Potapov product and $A \in H^\infty(\mathbb{T}, \mathcal{B}(E, D))$ is such that Δ and A are right coprime.

An answer to Question 6 (operator-valued extension of Potapov's Theorem).

Corollary

A two-sided inner function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is rational if and only if it can be represented as a finite Blaschke-Potapov product.

Thank you all for listening!