# Local theory

for

# stable polynomials

with applications to

# integrability for rational functions of several variables

Joint work with

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**Def** $p \in \mathbb{C}[z_1, \dots, z_d]$

is stable in domain $\Omega \subset \mathbb{C}^d$

if

$$p(z) \neq 0 \quad \text{in } \Omega$$

**Ex**: $p = 2 - z_1 - z_2$ stable

in $\{|z_1| < 1, |z_2| < 1\}$

**Def**: $f = \dfrac{q}{p}$ $\left(p, q \in \mathbb{C}[z_1, \dots, z_d]\right)$

is bounded rational function

in $\Omega$ if $f$ analytic in $\Omega$

and

$$|f(z)| < M, \quad z \in \Omega.$$

**Ex**: $$f = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$$

has $|f(z)| < 1$,

for $z \in \mathbb{D}^2 = \{|z_j| < 1, j = 1, 2\}$

▷ Stable polynomials & denominators of bdd rat. fns.

# Boundedness

## Basic Q:

given a stable polynomial $p$, which choices of $q$ make

$$\frac{q}{p} \text{ bounded ?}$$

- Typically, we consider

polydisk

$$\Omega = \mathbb{D}^d = \mathbb{D} \times \cdots \times \mathbb{D}$$

or poly-upper half-plane

$$\Omega = \mathbb{H}^d = \mathbb{H} \times \cdots \times \mathbb{H}$$

- If $p(z) \neq 0$ for $z \in \overline{\Omega}$

then $\frac{q}{p}$ bdd for any poly $q$.

What if $p(z) = 0$ for some $z \in \partial\Omega$?

$d = 1$ : $f = \frac{q}{p}$ bdd

if & only if

$$p = (z - z_0)^N \cdot P(z)$$

implies

$$q = (z - z_0)^M Q(z)$$

w/ $M \geq N$.

$d \geq 2$ : subtler & more interesting!

- cannot "standardize" zeros on $\overline{\Omega}$, so given $p$ stable, not clear when $\frac{q}{p}$ bdd.

- if $\frac{q}{p}$ is bdd, any additional regularity?

Eg., is

$$\partial_{z_1}\left(\frac{q}{p}\right) \in L^p ?$$

**Example:** Is

$$f_1 = \frac{z_1 - 1}{2 - z_1 - z_2}$$

bdd in $\mathbb{D}^2$? (No.)

Is

$$f_2 = \frac{(z_1 - 1)(z_2 - 1)}{2 - z_1 - z_2}$$

bdd in $\mathbb{D}^2$? (yes.)

- All polys above vanish at $(1,1) \in \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$

unit circle.

- Behavior near distinguished boundary is critical: When $\Omega = \mathbb{D}^2$, focus on 2-torus

$$\mathbb{T}^2 = \{ |z_j| = 1, \ j = 1, 2 \}$$

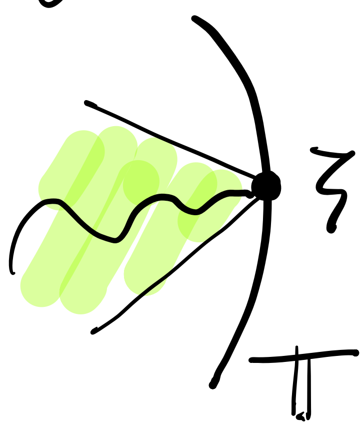
**Thm** (BKPS '21)

Sps $f = \frac{q}{p}$ is bdd rat. fcn.

Then the non-tangential limit

$$f^*(\zeta) = \angle \lim_{z \to \zeta} f(z)$$

exists for every $\zeta \in \mathbb{T}^d$.




▷ Fatou thm guarantees $f^*$ a.e.

▷ Earlier result for rational inner functions

bdd rat. w/

$(|f^*(\zeta)| = 1 \text{ a.e.})$

due to Knese '15.

**Note:** Bdd $f = \frac{q}{p}$ is typically

not continuous on $\overline{\Omega}$.

**Ex.:** $\left| \frac{(z_1 - 1)(z_2 - 1)}{2 - z_1 - z_2} \right| = 1$, for $z_2 = \overline{z_1}$ on $\mathbb{T}^2$

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Given $p$, any obvious admissible numerators?

i) $q = r \cdot p$, $r \in \mathbb{C}[z_1, \dots, z_d]$

$\Rightarrow f = r$, poly

ii) $q = r \cdot \tilde{p}$

where $\tilde{p}$ reflection of $p$

In $\Omega = \mathbb{C}^d$: $\tilde{p}(z) = z^{n} \overline{p}\left(\frac{1}{\overline{z}}\right)$ $\qquad z_1^{n_1} \cdots z_d^{n_d}$

$\Rightarrow f = r \frac{\tilde{p}}{p}$ $\leftarrow$ modulus 1 on $\mathbb{T}^d$

iii) $q \in (p, \tilde{p})$ ideal generated by $p$ & $\tilde{p}$.

Is this it?

Rest of talk: $d = 2$

$\Omega = \mathbb{D}^2$ or $\mathbb{H}^2$

▷ Can assume (b/c of Bézout)

$Z(p) \cap \mathbb{T}^2$ finite

zero set of poly $p$

$2^d$ is special

"Pure stable case"

▷ Can assume boundary zero is at $(1,1) \in \mathbb{T}^2$.

(Local Q)

(initially)

**Thm** (BKPY '21)

Sps $p$ generic pure stable.
(vanishes to order 1)

Then: $\frac{q}{p}$ bdd near $(1,1)$

$\Longleftrightarrow$

$q \in (p, \tilde{p})$ ideal generated by $p$ & $\tilde{p}$

▷ Previous example had $q = \frac{1}{2}(p + \tilde{p})$
$= (1-z_1)(1-z_2)$

▷ Difficulty lies in showing that boundedness of $\frac{q}{p}$ implies $q$ belongs to specific ideals.

▷ Situation more complicated when $p$ vanishes to higher order.

▷ Conjecture for general case ◁
(stated later)

## Integrability:

### Basic Q:

If $\frac{q}{p}$ is bounded, for which $t > 0$ is
$$\partial_{z_1}\left(\frac{q}{p}\right) \in L^t(\mathbb{T}^2)\ ?$$

- A measure of regularity of $\frac{q}{p}$.
- Can ask $L^t_{loc}$ version of Q.

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**Ex:** $\phi = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$ has

$$\partial_{z_1}\phi = -2\,\frac{(z_2-1)^2}{(2-z_1-z_2)^2} \in L^t(\mathbb{T}^2)$$
$$\iff t < \frac{3}{2}.$$

Also: $q = (z_1 - 1)\tilde{p}$ has

$$\partial_{z_1}\left(\frac{q}{p}\right) \in L^t(\mathbb{T}^2)$$

$$\iff t < 3$$

and $q = (z_1 - 1)^2 \tilde{p}$ has

$$\partial_{z_1}\left(\frac{q}{p}\right) \text{ bounded.}$$

▷ These are the **integrability indices** for derivatives of bdd $\frac{q}{p}$ for $p$ stable.

**Def:** Say $T > 0$ is $z_1$-derivative integrability index of $p$ if there $q$ s.t. $\frac{q}{p}$ bdd and $T = \sup\left\{t : \partial_{z_1}\left(\frac{q}{p}\right) \in L^t(\mathbb{T}^2)\right\}$.

**Thm** (BKPS '21)

Sps $p$ is pure stable, with zeros

$\tau_1, \ldots, \tau_m \in \mathbb{T}^2$.

If $p$ vanishes to order 1 at each zero $\tau_j$, then the $z_1$-derivative indices of $\frac{q}{p}$ belong to a finite list

$$\frac{K_j+1}{K_j}, \frac{K_j+1}{K_j-1}, \frac{K_j+1}{K_j-2}, \ldots, \frac{K_j+1}{1}, \infty$$

$$(j = 1, \ldots, m)$$

where the $K_j$ are the contact orders of $p$ at the $\tau_j$'s.

▷ A local algebraic characteristic of $p$.

**Ex:**

$p = 2 - z_1 - z$

has

$\kappa_1 = 2$ (smallest possible)

so get

$\frac{2+1}{2} = \frac{3}{2}$

and

$\frac{2+1}{2-1} = 3$

and

$\infty$.

What is contact order?

## Local theory:

flatten $\mathbb{D}^2$ and $\mathbb{T}^2$

to

$\mathbb{H}^2 = \{ \operatorname{Im} z_j > 0 \}$

and $\mathbb{R}^2$

map $(1,1)$ to $(0,0)$

Thm: pure stable $p$ can be factored as

$$p = u \cdot p_1 \cdot \ldots \cdot p_k$$

unit ↑ Weierstraß factor

▷ The $p_j$ take on a special form if $p$ pure stable.

Namely: each

$p_j(z_1, z_2)$

$$= \prod_{m=1}^{M_j} \left( z_2 + q_j(z_1) + z_1^{2L_j} \psi_m(z_1) \right)$$

where

- $q_j \in \mathbb{R}[z_1]$ w/ $\begin{cases} q_j(0) = 0 \\ q_j'(0) > 0 \end{cases}$

  $\deg q_j < 2L_j$

- $L_j \in \mathbb{N}$
- $\psi$ anal. near $0$, $\operatorname{Im} \psi(0) > 0$

  and $\psi_m(z_1) = \psi\left(e^{2\pi i \frac{m}{M_j}} z_1^{\frac{1}{M_j}}\right)$

($\triangleright$ Can happen that $M_j > 1$.)

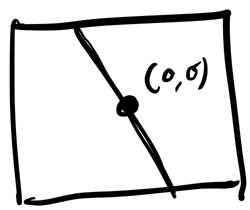
**Special case:** order of vanishing 1 at $(0,0)$

$$p = u \cdot p_1$$

where

$$p_1(z_1, z_2) = z_2 + q_1(z_1) + z_1^{2L}\psi(z_1)$$

- ▷ $K = 2L$ is contact order (▷ can be $> 2$.)
- ▷ $z_2 + q_1(z_1) = 0$ gives a special curve on which we examine $\frac{q}{p}$.




- ▷ When order of vanishing higher: additional combinatorial complexity.

## General case:

Define **product ideal**

$$J = \prod_{j=1}^{k} \left( z_2 + q_j(z_1),\ z_1^{2l_j} \right)^{M_j}$$

▷ generally, $J$ larger than $(p, \tilde{p})$.

**Conjecture:** $\frac{q}{p}$ bdd near $(0,0)$

$\iff$

$q \in J$.

**Thm:** (BKPS '21)

True if order of vanishing is 2.

True if ordinary multiple pt.

Thank

You !