Transformations of structures on positive definite cones in C^*-algebras

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Abstract:

In this talk we give a survey of our recent work on different 'symmetries' of (algebraic and geometric) structures on positive definite cones in C^*-algebras. Among others, we present precise descriptions of various algebraic isomorphisms, mean preservers, surjective isometries, etc.

Plan of the talk:

- Jordan *-isomorphisms of C^*-algebras;
- algebraic isomorphisms of positive definite cones;
- mean preservers;
- isometries, "generalized" isometries;
- preservers of norms of means.

Key concept is that of the Jordan *-isomorphisms.

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. The bijective linear map $J : \mathcal{A} \to \mathcal{B}$ is called a Jordan *-isomorphism (also called C^*-isomorphism) if it satisfies

$$J(A^2) = J(A)^2, \quad A \in \mathcal{A},$$

or, equivalently,

$$J(AB + BA) = J(A)J(B) + J(B)J(A), \quad A, B \in \mathcal{A},$$

and

$$J(A^*) = J(A)^*, \quad A \in \mathcal{A}.$$

Jordan *-isomorphisms are the most fundamental symmetries between C^*-algebras. They have important preserver properties.

Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras. If $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism, then

- J is a surjective linear isometry;
- J is an order isomorphism, i.e.,

$$A \leq B \iff J(A) \leq J(B), \quad A, B \in \mathcal{A}_s; \tag{1}$$

- J preserves commutativity in both directions, i.e.,

$$AB = BA \iff J(A)J(B) = J(B)J(A), \quad A, B \in \mathcal{A}; \tag{2}$$

- J preserves the projections in both directions, i.e.,

$$P \in \mathcal{A} \text{ is a projection} \iff J(P) \text{ is a projection}; \tag{3}$$

- J preserves the unitaries in both directions;

- J preserves invertibility in both directions and, in fact, we have $J(A^{-1}) = J(A)^{-1}$ for any invertible $A \in \mathcal{A}$. Moreover, J preserves the spectrum,
$$\sigma(J(A)) = \sigma(A), \quad A \in \mathcal{A}; \tag{4}$$
- J is compatible with the continuous functional calculus:
$$J(f(A)) = f(J(A)) \tag{5}$$
for any self-adjoint $A \in \mathcal{A}$ and continuous function f on its spectrum;
- etc.

There are many results in the literature which say that, in many cases, the above preserver properties characterize Jordan *-isomorphisms to certain extents within the collection of linear transformations.

Various structures on positive definite matrices appear in different areas of mathematics, physics, engineering (optimization, quantum information science, machine learning, diffusion tensor imaging, etc).

Our main interest is in the various (algebraic and geometric) structures on positive definite cones in the general setting of C^*-algebras, and in the descriptions of the corresponding (non-linear) automorphisms/isomorphisms, symmetries.
We will see that in many cases those transformations are closely related to the (linear) Jordan *-isomorphisms of the underlying full algebras.

Notation:

$\mathcal{A}$: unital C^*-algebra
$\mathcal{A}_s$: space of all self-adjoint elements
$\mathcal{A}^+$: positive semidefinite cone
$\mathcal{A}^{++}$: positive definite cone

Operations on positive definite cones and the corresponding isomorphisms.

The additive structure

Theorem 1 (2015)

*Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and assume that $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a bijective additive map. Then there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}^{++}$ such that*

$$\phi(A) = TJ(A)T, \quad A \in \mathcal{A}^{++}.$$

Multiplicative structure

If $A, B \in \mathcal{A}^{++}$, then ABA is called the Jordan triple product of A and B while $AB^{-1}A$ is said to be their inverted Jordan triple product (this will play an important role later).

If $\mathcal{B}$ is another C^*-algebra and $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a bijective map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathcal{A}^{++},$$

then ϕ is called a Jordan triple isomorphism. No linearity is assumed!

If the bijective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ fulfills

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathcal{A}^{++},$$

then ϕ is said to be an inverted Jordan triple isomorphism.

Easy transition from inverted Jordan triple isomorphisms to Jordan triple isomorphisms: $\phi(.) \to \phi(1)^{-1/2}\phi(.)\phi(1)^{-1/2}$.

What are the Jordan triple isomorphisms between positive definite cones?

A linear functional $l : \mathcal{A} \to \mathbb{C}$ is said to be tracial if it has the property that $l(AB) = l(BA)$, $A, B \in \mathcal{A}$.

Tr denotes the unique normalized tracial positive linear functional on finite factors.

Theorem 2 (2015)

*Assume $\mathcal{A}, \mathcal{B}$ are von Neumann algebras and $\mathcal{A}$ is a factor not of type I_2. Let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a continuous Jordan triple isomorphism. Suppose $\mathcal{A}$ is of infinite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and $c \in \{-1, 1\}$ such that*

$$\phi(A) = \theta(A^c), \qquad A \in \mathcal{A}^{++}.$$

*Assume $\mathcal{A}$ is of finite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, $c \in \{-1, 1\}$, and a real number d with $d \neq -c$ such that*

$$\phi(A) = e^{d \operatorname{Tr}(\log A)} \theta(A^c), \qquad A \in \mathcal{A}^{++}.$$

The converse statements are also true. Hence, we have the precise structure of continuous Jordan triple isomorphisms between positive definite cones in von Neuman factors not of type I_2.

True for factors of type I_2, too. Somewhat surprisingly, the proof is quite difficult, joint work with D. Virosztek (2016). Applications: structure of the (continuous) automorphisms (actually, even endomorphisms) of the operation of the Einstein velocity addition.

What about the case of C^*-algebras? We do not know.

Other important operations on positive definite cones: means

Concept of Kubo-Ando means.
Let $f :]0, \infty[\to]0, \infty[$ be an operator monotone function with the property $f(1) = 1$. The corresponding Kubo-Ando mean σ on the positive definite cone of a C^*-algebra $\mathcal{A}$ is

$$A \sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \quad A, B \in \mathcal{A}^{++}.$$

(The most important properties of those means are the monotonicity in their variables and the transfer equality/property, i.e., invariance under all congruence transformations.)

The most distinguished Kubo-Ando means are naturally the arithmetic mean with representing function $t \mapsto (1 + t)/2$, the harmonic mean corresponding to $t \mapsto (2t)/(1 + t)$ and the geometric mean corresponding to $t \mapsto \sqrt{t}$, $t > 0$. For $A, B \in \mathcal{A}^{++}$, we respectively have

$$A \nabla B = \frac{A + B}{2}, \quad A!B = 2(A^{-1} + B^{-1})^{-1}, \quad A \sharp B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

Geometric mean as "geodesic midpoint".
Mean preservers as midpoint preservers.

Kubo-Ando power means, a parametric family connecting the three fundamental means.

For $p \in [-1, 1], p \neq 0$ the function $t \mapsto ((1 + t^p)/2)^{1/p}$ is operator monotone and we define the Kubo-Ando pth power mean $\mathfrak{m}_p$ as

$$A \mathfrak{m}_p B = A^{1/2} \left(\frac{I + (A^{-1/2} B A^{-1/2})^p}{2} \right)^{1/p} A^{1/2}, \quad A, B \in \mathcal{A}^{++}. \tag{6}$$

Moreover, if $p \to 0$, we have

$$A \mathfrak{m}_p B \to A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2} = A \sharp B$$

in norm. Therefore, $A \mathfrak{m}_0 B = A \sharp B$.
Apparently, $\mathfrak{m}_1 = \nabla$, $\mathfrak{m}_{-1} = !$.

So, $\mathfrak{m}_p$, $p \in [-1, 1]$ is a parametric family of Kubo-Ando means connecting the three fundamental means.

Isomorphisms under the Kubo-Ando power means

Theorem 3 (2020)

Let $p \in [-1, 1]$, $p \neq 0$, and let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a continuous bijective map. Then ϕ satisfies

$$\phi(A\mathfrak{m}_p B) = \phi(A)\mathfrak{m}_p\phi(B), \quad A, B \in \mathcal{A}^{++}$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $T \in \mathcal{B}^{++}$ such that*

$$\phi(A) = TJ(A)T, \quad A \in \mathcal{A}^{++}.$$

The proof is based on a structural result concerning positive homogeneous order isomorphisms between positive definite cones which, surprisingly, follows from our description of Thompson isometries of positive definite cones to be considered in a few minutes.

What about geometric mean preservers (the $p = 0$ case)?

To describe them we recall Anderson-Trapp theorem saying that the unique solution of the equation $XA^{-1}X = B$ on a positive definite cone is exactly the geometric mean $A \sharp B$.

We obtain that geometric mean preservers = inverted Jordan triple isomorphims. Furthermore, inverted Jordan triple isomorphisms are closely related to Jordan triple isomorphisms which we have already determined in the case of von Neumann factors. Hence, we obtain the following.

Theorem 4 (2015)

Assume $\mathcal{A}, \mathcal{B}$ are von Neumann algebras, $\mathcal{A}$ is a factor, $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a continuous bijective map which preserves the geometric mean.
*Suppose $\mathcal{A}$ is of infinite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, $c \in \{-1, 1\}$, and an element $T \in \mathcal{B}^{++}$ such that*

$$\phi(A) = T\theta(A^c)T, \qquad A \in \mathcal{A}^{++}.$$

*Assume $\mathcal{A}$ is of finite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, $c \in \{-1, 1\}$, a real number d with $d \neq -c$, and an element $T \in \mathcal{B}^{++}$ such that*

$$\phi(A) = e^{d \operatorname{Tr}(\log A)} T\theta(A^c)T, \qquad A \in \mathcal{A}^{++}.$$

Converse statements are also true.

Natural question, difficult open problem: Structure of bijective continuous maps between positive definite cones which preserve a general Kubo-Ando mean.

What about nonbijective mean preservers, e.g., "functionals" of means?

Any nonzero positive linear functional f is a functional of the arithmetic mean. The functional $A \mapsto f(A^{-1})^{-1}$ is a functional of the harmonic mean.

If $p = 0$, we have the following.

Theorem 5 (2021)

Let $\mathcal{A}$ be a von Neumann algebra. Let $F : \mathcal{A}^{++} \to]0, \infty[$ be a continuous function. We have

$$F(A\mathfrak{m}_0 B) = F(A\sharp B) = \sqrt{F(A)F(B)}, \quad A, B \in \mathcal{A}^{++} \tag{7}$$

if and only if there is a tracial bounded linear functional $l : \mathcal{A} \to \mathbb{C}$ which is real valued on $\mathcal{A}_s$, and a positive real number c such that

$$F(A) = c \exp(l(\log(A))), \quad A \in \mathcal{A}^{++}.$$

What about the other Kubo-Ando means $\mathfrak{m}_p$, where $-1 < p < 1$, $p \neq 0$?

Theorem 6 (2021)

Let $\mathcal{A}$ be a von Neumann algebra without type I_2, I_1 direct summands. If $p \in]-1, 1[$, $p \neq 0$, then any function $F : \mathcal{A}^{++} \to]0, \infty[$ satisfying

$$F(A\mathfrak{m}_p B) = F(A)\mathfrak{m}_p F(B), \quad A, B \in \mathcal{A}^{++}$$

is necessarily constant.

Possibility of a joint characterization of the three fundamental means?

For example, is it true that the only symmetric Kubo-Ando means on the positive definite cones of matrix algebras which have nonconstant functionals are the arithmetic, geometric and harmonic means?

Thompson metric (or Thompson part metric): Let $\mathcal{A}$ be a C^*-algebra. The metric d_T on $\mathcal{A}^{++}$ is defined as follows:

$$d_T(A, B) = \log\max\{M(A/B), M(B/A)\}, \quad A, B \in \mathcal{A}^{++},$$

where $M(X/Y) = \inf\{t > 0 : X \leq tY\}$ for any $X, Y \in \mathcal{A}^{++}$. It is easy to see that d_T can also be rewritten as

$$d_T(A, B) = \left\|\log\left(A^{-1/2}BA^{-1/2}\right)\right\|, \quad A, B \in \mathcal{A}^{++}.$$

d_T is a shortest path distance in an appropriate Finsler-type geometry on $\mathcal{A}^{++}$ which has a number of applications in several areas.

The key property of d_T is that it makes the positive definite cone a complete(!) metric space while its topology coincides with the topology of the C^*-norm.

Description of Thompson isometries:

Observe $A \mapsto J(A)$, $A \mapsto A^{-1}$, $A \mapsto TAT$ (where J is a Jordan *-isomorphism, T is a fixed element of $\mathcal{A}^{++}$) are all Thompson isometries.

Theorem 7 (O. Hatori and M, 2014)

*The surjective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a Thompson isometry iff there is a central projection P in $\mathcal{B}$ and a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ such that ϕ is of the form*

$$\phi(A) = \phi(1)^{1/2} \left(PJ(A) + (1-P)J(A^{-1})\right) \phi(1)^{1/2}, \quad A \in \mathcal{A}^{++}.$$

Fundamental steps in the proof:
We may assume that $\phi(I) = I$.

The classical Mazur-Ulam theorem says that surjective isometries between real normed linear spaces are affine, they preserve the operation of the arithmetic mean.
Mazur and Ulam's original argument can be applied (with modifications) to show the interesting fact that Thompson isometries necessarily preserve the operation of the geometric mean:

$$\phi(A\sharp B) = \phi(A)\sharp\phi(B) \qquad A, B \in \mathcal{A}^{++}.$$

As mentioned before, by Anderson-Trapp theorem, we have

$$\phi(XA^{-1}X) = \phi(X)\phi(A)^{-1}\phi(X), \quad A, X \in \mathcal{A}^{++}.$$

Since ϕ sends the unit to the unit, we easily obtain that ϕ satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathcal{A}^{++}.$$

We have also mentioned that the topologies of the Thompson metric and the C^*-norm coincide on the positive definite cone. Therefore, ϕ is continuous with respect to the norm topology, ϕ is a continuous Jordan triple isomorphism.

$\phi(A^t) = \phi(A)^t$ holds for all $A \in \mathcal{A}^{++}$ and $t \in \mathbb{R}$.

Using Lie-Trotter formula, it can be shown that the map $F : \mathcal{A}_s \to \mathcal{B}_s$ defined by $F(X) = \log \phi(e^X)$, $X \in \mathcal{A}_s$ is a bijective linear transformation.

Next, we use that our map is a Thompson isometry. On the one hand, we easily see that

$$\frac{d_T\left(e^{tX}, I\right)}{t} \xrightarrow{t \to 0} \|X\|.$$

Using the fact that ϕ is a Thompson isometry, we can deduce that

$$\|F(X)\| = \|X\|, \quad X, Y \in \mathcal{A}_s.$$

Hence, F is a surjective linear isometry between the self-adjoint parts of C^*-algebras.

A result of Kadison describes the structure of those transformations. There is a central self-adjoint unitary element S in $\mathcal{A}$ and a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$F(X) = SJ(X), \quad X \in \mathcal{A}_s.$$

Clearly, $S = 2P - I$ holds with some central projection $P \in \mathcal{B}$.

The rest of the proof of the necessity part of the result is now a simple calculation and so is its sufficiency part.

An important corollary (we call it THE COROLLARY) follows.

Corollary 8

Let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be an order isomorphism, i.e., a surjective mapping with the property that for any $A, B \in \mathcal{A}^{++}$ we have

$$A \leq B \iff \phi(A) \leq \phi(B).$$

If ϕ is also positive homogeneous (i.e., satisfies $\phi(\lambda A) = \lambda \phi(A)$ for all $A \in \mathcal{A}^{++}$ and positive real number λ), then ϕ is necessarily of the form

$$\phi(A) = TJ(A)T, \quad A \in \mathcal{A}^{++},$$

*where $T \in \mathcal{A}^{++}$ and $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism.*

Consequence: Homogeneous order isomorphisms are automatically additive!
Recall that the proof of our result on the structure of isomorphisms with respect to KA power means is based on this important observation.
We have a number of other applications, too.

Remark: There are important recent results by Mori on automatic linearity of order isomorphisms (no homogeneity is assumed!) in von Neumann algebras without commutative direct summands.

In the setting of von Neumann algebras, we can generalize the structural result on Thompson isometries as follows.

Recall the formula for d_T:

$$d_T(A,B) = \left\| \log \left(A^{-1/2} B A^{-1/2} \right) \right\|, \quad A, B \in \mathcal{A}^{++}.$$

Consider so-called generalized distance measures of the following form:

$$d_{N,f} : (A,B) \mapsto N(f(A^{-1/2} B A^{-1/2})),$$

where N is a norm on $\mathcal{A}$ and $f :]0,\infty[\to \mathbb{R}$ is a continuous function which takes the value 0 only at 1.

$d_{N,f}$ is surely not a metric in general. We only have that it is nonnegative and $d_{N,f}(A,B) = 0$ iff $A = B$ (this is what we mean by a generalized distance measure, or divergence).

Examples for such distance measures (on matrix algebras) include: Stein's loss, symmetric Stein divergence, Jeffrey's Kullback–Leibler divergence, log-determinant α-divergence (they show up in optimization, statistics, quantum information).

I	$\|\|.\|\|_1$	$f(y) = y^{-1} - \log y^{-1} - 1,\ y > 0$
S_{JS}	$\|\|.\|\|_1$	$f(y) = \log((y+1)/(2\sqrt{y})),\ y > 0$
S_{JKL}	$\|\|.\|\|_1$	$f(y) = (y + y^{-1} - 2)/2,\ y > 0$
D^{α}_{LD}	$\|\|.\|\|_1$	$f(y) = \log\left(\frac{(1-\alpha)+(1+\alpha)y}{2y^{(1+\alpha)/2}}\right),\ y > 0$

Using our general result concerning Jordan triple isomorphisms between positive definite cones in von Neumann algebras we could determine the corresponding symmetries, i.e., maps preserving those generalized distance measures.

Bures metric and its isometries:

Farenick et al. in 2016/17 introduced the concepts of fidelity and Bures (or Bures-Wasserstein) distance (important notions in quantum information science) in the general setting of C^*-algebras carrying faithful traces.

If $\mathcal{A}$ is a C^*-algebra and τ is a positive linear functional on $\mathcal{A}$ which is tracial ($\tau(XY) = \tau(YX)$ holds for all $X, Y \in \mathcal{A}$), then we call τ a trace. We say that the trace τ on $\mathcal{A}$ is faithful if $\tau(A) = 0$, $A \in \mathcal{A}^+$ implies $A = 0$.

Examples for such algebras include finite factor von Neumann algebras, UHF-algebras, irrational rotation algebras, etc.

Let $\mathcal{A}$ be a C^*-algebra and τ be a faithful trace on $\mathcal{A}$. For any $A, B \in \mathcal{A}^+$, we define

$$F_\tau(A, B) = \tau\left((A^{1/2} B A^{1/2})^{1/2}\right)$$

and

$$d_B^\tau(A, B) = \sqrt{\tau(A) + \tau(B) - 2F_\tau(A, B)}$$

The first quantity is called the fidelity of A and B, the second one is the Bures (or Bures-Wasserstein) distance between A and B.

It can be shown that d_B^τ is a true metric on $\mathcal{A}^+$.

The next theorem describes the surjective Bures isometries between positive definite cones of C^*-algebras.

Theorem 9 (2018)

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, τ', respectively, and let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a surjective map. Then ϕ is a Bures-Wasserstein isometry, i.e., it satisfies

$$d_B^{\tau'}(\phi(A), \phi(B)) = d_B^{\tau}(A, B), \quad A, B \in \mathcal{A}^{++}$$

if and only if
*there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$,*
and an element $C \in \mathcal{B}^{++}$ central in $\mathcal{B}$ such that
$\phi(A) = CJ(A)$ holds for all $A \in \mathcal{A}^{++}$ and
$\tau'(CJ(X)) = \tau(X)$ holds for all $X \in \mathcal{A}$.

Main idea of the proof: we have a characterization of the order by the Bures metric:

For any $A, B \in \mathcal{A}^{++}$, consider the set

$$\{(d_B^\tau(B,X))^2 - (d_B^\tau(A,X))^2 \,:\, X \in \mathcal{A}^{++}\}.$$

It is bounded from above if and only if $A \leq B$.

We then deduce in turn:
ϕ preserves the order.
ϕ has limit 0 at 0.
ϕ preserves the trace, ϕ preserves the fidelity. ϕ is positive homogeneous.
ϕ is of the form $\phi(A) = CJ(A)C$, $A \in \mathcal{A}^{++}$ with some $C \in \mathcal{B}^{++}$ and Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$. Application of THE COROLLARY.
ϕ preserves the trace $\Rightarrow$ C is central.

Motivation from two directions:

Norm additive maps: first studied on function algebras by T. Tonev and his group. The aim was to present conditions for nonlinear maps to be necessarily weighted composition operators on function algebras.
The norm additivity property is equivalent to the preservation of the norm of the arithmetic mean.

A recent investigation of certain quantum symmetries: study of maps on so-called density spaces of C^*-algebras which preserve different sorts of quantum Rényi relative entropies (traces of variants of the geometric mean).

Theorem 10 (Dong, Li, M, Wong (2021?))

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. Assume that $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a bijective map. Then ϕ satisfies

$$\|\phi(A) + \phi(B)\| = \|A + B\|, \quad A, B \in \mathcal{A}^{++}, \tag{8}$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ which extends ϕ, i.e., $\phi(A) = J(A)$ holds for all $A \in \mathcal{A}^{++}$.*

The proof is surprisingly difficult and complicated.

Concerning the preservers of the norm of the geometric mean we have the same description.

Theorem 11 (Chabbabi, Mbekhta, M (2019))

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and assume that $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a bijective map. It satisfies

$$\|\phi(A)\sharp\phi(B)\| = \|A\sharp B\|, \quad A, B \in \mathcal{A}^{++}$$

*if and only if there is Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ which extends ϕ.*

Next, the result concerning the harmonic mean:

Theorem 12 (Dong, Li, M, Wong (2021?))

Let $\mathcal{A}, \mathcal{B}$ be AW^*-algebras. Assume that $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a surjective map. Then ϕ satisfies

$$\|\phi(A)!\phi(B)\| = \|A!B\|, \quad A, B \in \mathcal{A}^{++},$$

if and only if it extends to a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

How to prove those statements? Again, the strategy is to prove that the maps in question are necessarily positive homogeneous order isomorphisms and then apply our result on the structure of such maps (THE COROLLARY).

Theorem 13 (2019-20)

For any $A, B \in \mathcal{A}^{++}$ we have
$A \leq B$ *if and only if* $\|A \nabla X\| \leq \|B \nabla X\|$ *holds for all* $X \in \mathcal{A}^{++}$;
$A \leq B$ *if and only if* $\|A \sharp X\| \leq \|B \sharp X\|$ *for all* $X \in \mathcal{A}^{++}$
$A \leq B$ *if and only if* $\|A ! X\| \leq \|B ! X\|$, $X \in \mathcal{A}^{++}$.

We currently study this problem for general Kubo Ando means, namely, that which of them have the property that via their norms the order can be determined.

There are several other types of means: Fiedler-Pták spectral geometric mean, Wasserstein mean, conventional power means just to name a few. Some of the problems similar to the ones above related to them are really exciting and challenging.

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Thank you very much for your kind attention!