Nonlocal Douglas identity in L^p

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arXiv 2019/JMPA 2020 [4].
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Classical Hardy-Stein and Douglas formulas

Let $D = \{z \in \mathbf{R}^2 : |z| < 1\}$, $T = \partial D \sim [0, 2\pi)$. Let

$$u(z) = \int_T g(\theta) P_D(z, \theta) \mathrm{d}\theta$$

(harmonic function), where

$$P_D(z, \theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - \theta|^2}.$$

Then (Hardy-Stein identity)

$$\frac{1}{2\pi} \int_T |g(\theta)|^2 \mathrm{d}\theta = |u(0)|^2 + 2 \int_D G_D(0, z) |\nabla u(z)|^2 \mathrm{d}z.$$

Also (J. Douglas 1931),

$$\int_D |\nabla u(z)|^2 \mathrm{d}z = \iint_{T \times T} (g(\theta) - g(\eta))^2 \frac{1}{8\pi} \frac{1}{\sin^2((\theta - \eta)/2)} \mathrm{d}\theta \mathrm{d}\eta.$$

The nonlocal (unimodal Lévy) setting of [4]

Let $\nu\colon [0,\infty) \to (0,\infty]$ be nonincreasing and $d \in \{1,2,\ldots\}$. Denote $\nu(z) = \nu(|z|)$, $z \in \mathbb{R}^d$. Assume $\int_{\mathbb{R}^d} \nu(z)\mathrm{d}z = \infty$ and

$$\int_{\mathbb{R}^d} \left(|z|^2 \wedge 1\right) \nu(z)\mathrm{d}z < \infty.$$

Denote $\nu(x,y) = \nu(y-x)$. For $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$ let

$$Lu(x) \;=\; \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} (u(y)-u(x))\nu(x,y)\,\mathrm{d}y.$$

The limit exists, e.g., for $u \in C_c^2(\mathbb{R}^d)$. If $0 < \alpha < 2$ and

$$\nu(z) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}|z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,$$

then L is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$.

Sobolev space

Let $D \subset \mathbb{R}^d$ be open, nonempty and Lipschitz. After Servadei and Valdinoci [24], Felsinger, Kassmann and Voigt [12] and Millot, Sire and Wang [20] for $u : \mathbb{R}^d \to \mathbb{R}$ we consider

$$\mathcal{E}_D(u,u) = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \nu(x,y) \mathrm{d}x\mathrm{d}y,$$

cf. Rutkowski [22], Ros-Oton [21]. Also, $\mathcal{E}_D(u,v) := \ldots$. Let

$$\mathcal{V}^D = \{u : \mathbb{R}^d \to \mathbb{R} \text{ such that } \mathcal{E}_D(u,u) < \infty\},$$

a Sobolev-type space. Furthermore,

$$\mathcal{V}_0^D := \{u \in \mathcal{V}^D : \ u = 0 \ a.e. \text{ on } D^c\}.$$

Recall the usual Dirichlet form of L:

$$\mathcal{E}_{\mathbb{R}^d}(u,u) = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 \nu(x,y) \, \mathrm{d}x\mathrm{d}y.$$

Weak Dirichlet problem

Then, $\mathcal{E}_{\mathbb{R}^d}(u,u) = \mathcal{E}_D(u,u) + \frac{1}{2}\iint_{D^c \times D^c} (u(x) - u(y))^2 \nu(x,y)\, \mathrm{d}x\mathrm{d}y$, and

$\mathcal{E}_D(u,u) = \mathcal{E}_{\mathbb{R}^d}(u,u)$ if $u = 0$ on D^c. If $\phi \in C_c^\infty(D)$, $u \in C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{E}_D(u,\phi) = \mathcal{E}_{\mathbb{R}^d}(u,\phi) = -\int_{\mathbb{R}^d} \phi(x) Lu(x) \mathrm{d}x.$$

The Sobolev space $\mathcal{V}^D$ is suitable for solving the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } D, \\ u = g & \text{on } D^c. \end{cases} \tag{1}$$

We mean the *weak solutions*: $u \in \mathcal{V}^D$ with $u = g$ *a.e.* on D^c, i.e., an *extension* of g, and such that for all $\phi \in \mathcal{V}_0^D$,

$$\mathcal{E}_D(u,\phi) = 0, \quad \phi \in \mathcal{V}_0^D.$$

Extension problem

Definition

We say that the extension problem for g, D and L has solution if the exterior condition g has an extension $u \in \mathcal{V}^D$.

If so, then the existence and uniqueness for the Dirichlet problem (1) follow from the Lax-Milgram theory, see [12, 22].

For $\Delta^{\alpha/2}$, Dyda and Kassmann [10] solve the extension problem if

$$g \in L^2_{loc}(D^c) \quad \text{and} \quad \int_{D^c}\int_{D^c} \frac{(g(z)-g(w))^2}{r(z,w)^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w < \infty,$$

where $r(z,w) := \delta_D(z)+|z-w|+\delta_D(w)$ and $\delta_D(z) := \operatorname{dist}(z, \partial D)$.

For "local" extension theorems see, e.g., Adams and Fournier [1].

Unimodal operators L

Here are the additional assumptions on $\nu\colon [0,\infty) \to (0,\infty]$:

A1 $\nu'' \in C(0,\infty)$ and $|\nu'(r)|, |\nu''(r)| \le c\nu(r)$ for $r > 1$.

A2 There is $\beta \in (0,2)$ such that

$$\begin{aligned} \nu(\lambda r) &\le c\lambda^{-d-\beta}\nu(r), \qquad && 0 < \lambda,\ r \le 1, \\ \nu(r) &\le c\nu(r+1), && r \ge 1. \end{aligned}$$

A3 There is $\alpha \in (0,2)$ such that

$$\nu(\lambda r) \ge c\lambda^{-d-\alpha}\nu(r), \qquad 0 < \lambda,\ r \le 1.$$

For instance, **A1**, **A2** and **A3** hold true if $L = \Delta^{\alpha/2}$, in which case $\nu(r) = cr^{-d-\alpha}$.

Hardy-Stein formula for L

Let $G_D(x,y)$ be the Green function and $\omega_D^x(\cdot)$ be the harmonic measure of D for L. We say $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic on D if

$$u(x) = \int u(y)\omega_U^x(dy), \quad x \in U \subset\subset D.$$

Here is a Hardy-Stein formula for $L-$harmonic functions, as in Bogdan, Dyda, Luks [3], where it was given for $\Delta^{\alpha/2}$.

Lemma ([4])

If u is harmonic in D and $x \in U \subset\subset D$, then

$$\int_{U^c} u^2(y)\,\omega_U^x(\mathrm{d}y) = u(x)^2 + \int_U G_U(x,y) \int_{\mathbb{R}^d} (u(z)-u(y))^2 \nu(z,y)\mathrm{d}z\mathrm{d}y.$$

Poisson kernel P_D and the interaction kernel γ_D

The Poisson kernel of D for L is $P_D(x,z) := \int_D G_D(x,y)\nu(y,z)\mathrm{d}y$. For $g : D^c \mapsto \mathbb{R}$ we let $P_D[g](x) = g(x)$ for $x \in D^c$ and

$$P_D[g](x) = \int_{D^c} g(y) P_D(x,y)\mathrm{d}y \quad \text{for} \quad x \in D,$$

if finite. The interaction kernel for $w, z \in D^c$ is

$$\gamma_D(w,z) := \int_D \int_D \nu(w,x) G_D(x,y) \nu(y,z)\,\mathrm{d}x\mathrm{d}y.$$

Example

Let $d = 1$, $D = (0,\infty) \subset \mathbb{R}$ and $\nu(w,x) = \pi^{-1}|x-w|^{-2}$, i.e. $L = \Delta^{1/2}$. Then $P_{(0,\infty)}(x,z) = \pi^{-1}x^{1/2}|z|^{-1/2}(x-z)^{-1}$, and

$$\gamma_{(0,\infty)}(z,w) = \left(2\pi\sqrt{zw}\left(\sqrt{|z|}+\sqrt{|w|}\right)^2\right)^{-1}, \qquad z,w<0.$$

Douglas-type formula for L

Back to general L, for $g : D^c \to \mathbb{R}$ we let

$$\mathcal{H}_D(g,g) = \frac{1}{2} \iint\limits_{D^c \times D^c} (g(w) - g(z))^2 \gamma_D(w,z)\, \mathrm{d}w\mathrm{d}z.$$

Accordingly we define ($\mathcal{X}$ stands for e$\mathcal{X}$terior),

$$\mathcal{X}^D = \{g : D^c \to \mathbb{R} : \ \mathcal{H}_\mathcal{D}(g,g) < \infty\}.$$

Theorem (Nonlocal Douglas formula [4])

If $u = P_D[g]$, then $\mathcal{E}_D(u,u) = \mathcal{H}_D(g,g)$.

Thus, $$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \nu(x,y) = \iint\limits_{D^c \times D^c} (g(w) - g(z))^2 \gamma_D(w,z).$$

In the language of Chen and Fukushima [8], the right-hand is the *trace form* and $\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w$ is the *Feller measure*. Note that the extension and trace problem for $\mathcal{V}^D$ were not investigated in [8].

We prove Dirichlet-form regularity, *extension and trace theorems*. Note that $\mathcal{H}_D(g,g)$ may be finite for rather rough functions on D^c:

$$\mathcal{H}_D(g,g) = \tfrac{1}{2}\int_{D^c}\int_{D^c}(g(z)-g(w))^2\gamma_D(z,w) \leq \ 2\int_{D^c}\int_{D^c} g^2(z)\gamma_D(z,w)$$
$$= 2\int_{D^c}\int_{D^c} g^2(z)\int_D \nu(z,x)P_D(x,w) = 2\int_{D^c} g^2(z)\nu(z,D).$$

Example

As in the previous Example we let $u(x) = g(x)$ for $x \leq 0$, and

$$u(x) := P_D[g](x) = \int_{-\infty}^{0} \frac{\sqrt{x}g(z)\mathrm{d}z}{\pi(x-z)\sqrt{|z|}} \quad \text{for } x > 0.$$

If the above integral is absolutely convergent, then

$$\iint\limits_{x>0 \text{ or } y>0} \frac{(u(x)-u(y))^2}{\pi(x-y)^2} = \iint\limits_{z<0 \text{ and } w<0} \frac{(g(z)-g(w))^2}{2\pi\sqrt{zw}(\sqrt{|z|}+\sqrt{|w|})^2}.$$

Pruitt functions

For $r > 0$ we let

$$h(r) = \int_{\mathbb{R}^d} \left(\frac{|z|^2}{r^2} \wedge 1 \right) \nu(z) \mathrm{d}z,$$
$$V(r) = \frac{1}{\sqrt{h(r)}}.$$

Example

For $\Delta^{\alpha/2}$ we have $h(r) = cr^{-\alpha}$ and $V(r) = cr^{\alpha/2}$.

Bogdan, Grzywny, Ryznar [7, 6] estimate V for many ν's. For the next example recall the notation:

$$r(z,w) = \delta_D(z) + |z-w| + \delta_D(w) \quad \text{and} \quad \delta_D(z) = \operatorname{dist}(z, \partial D).$$

Estimates of γ_D for bounded $C^{1,1}$ open sets $D \subset \mathbb{R}^d$

Theorem

$$\gamma_D(z,w) \approx \begin{cases} \nu(\delta_D(w))\,\nu(\delta_D(z)) & \textit{if} \quad \delta_D(z), \delta_D(w) \geq \operatorname{diam}(D), \\ \nu(\delta_D(w))\,/\,V(\delta_D(z)) & \textit{if} \quad \delta_D(z) < \operatorname{diam}(D) \leq \delta_D(w), \\ \nu(r(z,w))\,V^2(r(z,w))\,/\,[V(\delta_D(z))\,V(\delta_D(w))] & \textit{else.} \end{cases}$$

Example

For $L = \Delta^{\alpha/2}$, the unit ball $D = B(0,1) \subset \mathbb{R}^d$, and $z, w \in D^c$,

$\gamma_D(z,w) \approx |w|^{-d-\alpha}|z|^{-d-\alpha}$ if both z and w are far from D,

$\approx |w|^{-d-\alpha}(|z|-1)^{-\alpha/2}$ if only w is far from D, else

$\approx \left((|w|-1) + |w-z| + (|z|-1)\right)^{-d}(|z|-1)^{-\alpha/2}(|w|-1)^{-\alpha/2}$.

Tools: the Lévy process and the first exit time

We define $\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot x)\, \nu(|x|)\, \mathrm{d}x,\ \xi \in \mathbb{R}^d$.
For $t > 0$ there is $p_t \geq 0$ such that

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi \cdot x} p_t(x)\, \mathrm{d}x = \mathrm{e}^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

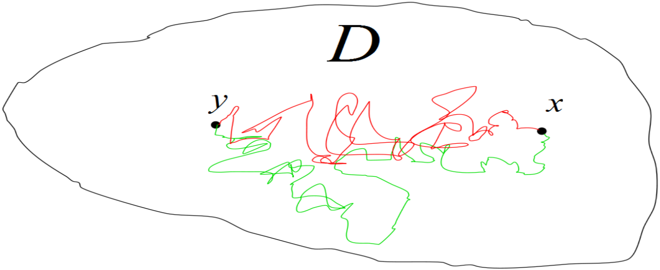
Let $X = \{X_t\}_{t \geq 0}$ be the symmetric Lévy process in $\mathbb{R}^d$ with the Lévy triplet $(0, \nu, 0)$ and transition density $p_t(x - y)$, cf. Sato [23].

Let $D \subset \mathbb{R}^d$ be open. The first exit time of D is:

$$\tau_D := \inf\{t > 0 : X_t \notin D\}.$$

The heat kernel

$$p_t^D(x,y) := p_t(y-x) - \mathbb{E}^x\left[\tau_D \le t;\ p_{t-\tau_D}(y - X_{\tau_D})\right]$$
$$p_t^D(x,y) := p_t(y-x) - \mathbb{E}^x\left[\tau_D \le t;\ p_{t-\tau_D}(y - X_{\tau_D})\right]$$



The cousins and relatives of the heat kernel

Connection to "killing at τ_D":

$$P_t^D f(x) := \mathbb{E}^x[t < \tau_D;\, f(X_t)] = \int_{\mathbb{R}^d} f(y) p_t^D(x,y)\mathrm{d}y,$$

The Green function of D is

$$G_D(x,y) := \int_0^\infty p_t^D(x,y)\mathrm{d}t.$$

Recall the Poisson kernel of D:

$$P_D(x,z) := \int_D G_D(x,y)\nu(y,z)\mathrm{d}y.$$

If D is Lipschitz, then $\mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$ for $x \in D$, and

$$\mathbb{P}^x(X_{\tau_D} \in B) = \int_B P_D(x,y)\mathrm{d}y, \quad x \in D, \quad B \subseteq D^c.$$

Ikeda-Watanabe and Dynkin formulas

Ikeda-Watanabe formula: for $J \subset \mathbb{R}$, $A \subset D$, $B \subset (\overline{D})^c$,

$$\mathbb{P}^x[\tau_D \in J, X_{\tau_D -} \in A, X_{\tau_D} \in B] = \int_J \int_B \int_A p_u^D(x,y)\nu(y,z)\mathrm{d}y\mathrm{d}z\mathrm{d}u.$$

I-W gives the law of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$ on $\{X_{\tau_D -} \in D\}$.

Consider nice $U \subset\subset D$ and $\phi : \mathbb{R}^d \to \mathbb{R}$, say, C^2. Then for $x \in U$,

$$\begin{aligned}\int \phi(y)\omega_U^x(\mathrm{d}y) &= \int_D \phi(z)P_D(x,z)\mathrm{d}z = \mathbb{E}^x\phi(X_{\tau_U}) \\ &\overset{Dynkin}{=} \phi(x) + \mathbb{E}^x\int_0^{\tau_U} L\phi(X_t)\mathrm{d}t = \phi(x) + \int_U G_U(x,y)L\phi(y)\mathrm{d}y\,.\end{aligned}$$

Hardy-Stein formula (explanation)

Recal that $u : \mathbb{R}^d \to \mathbb{R}$ is L-harmonic in D if for all open $U \subset\subset D$,

$$u(x) = \mathbb{E}^x u(X_{\tau_U}), \quad x \in U.$$

Using ν'' and Grzywny and Kwaśnicki [13] we get

Lemma

If u is L-harmonic on D, then $u \in C^2(D)$ and $Lu = 0$ on D.

Clearly, $b^2 - a^2 - 2a(b-a) = (b-a)^2$. If u is L-harmonic,

$$Lu^2(y) = Lu^2(y) - 2u(y)Lu(y) = \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z, y)\, \mathrm{d}z,$$

for $y \in U$. Applying Dynkin to $u(x)^2$, we get Hardy-Stein:

$$\mathbb{E}^x u(X_{\tau_U})^2 = u(x)^2 + \int_U G_U(x, y) \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z, y) \mathrm{d}z\mathrm{d}y.$$

Douglas formula (explanation)

Recall our Douglas-type formula: If $u = P_D[g]$, then

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))^2\nu(x,y) = \iint\limits_{D^c\times D^c} (g(w)-g(z))^2\gamma_D(w,z).$$

This is proved by Hardy-Stein, mysterious cancellations and

$$\mathbb{E}(Y-a)^2 = \mathrm{Var}(Y) + (\mathbb{E}Y-a)^2\,.$$

Corollary

$$\mathcal{E}_{\mathbb{R}^d}(u,u)=\frac{1}{2}\iint\limits_{D^c\times D^c}(u(w)-u(z))^2\left(\gamma_D(w,z)+\nu(w,z)\right)\,\mathrm{d}w\mathrm{d}z.$$

Bregman divergence [5]

For $x \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ we define (the French power)

$$x^{<\kappa>} = |x|^\kappa \operatorname{sgn}(x).$$

E.g., $x^{\langle 0 \rangle} = \operatorname{sgn}(x)$, $\sqrt[3]{x} = x^{\langle 1/3 \rangle}$ and $x^{\langle 2 \rangle} \neq x^2$ as functions on $\mathbb{R}$.
Note: $(|x|^\kappa)' = \kappa x^{<\kappa-1>}$ and $(x^{<\kappa>})' = \kappa |x|^{\kappa-1}$ for $x \neq 0$.

Let $p \in (1, \infty)$. Define, after Bogdan, Dyda, Luks [3],

$$F_p(a,b) = |b|^p - |a|^p - p a^{<p-1>}(b-a), \quad a, b \in \mathbb{R}.$$

$F_p(a,b)$ is an example of *Bregman divergence*, see Sprung [26].
E.g., $F_2(a,b) = (b-a)^2$ and $F_4(a,b) = (b-a)^2(b^2 + 2ab + 3a^2)$.

By convexity of $|x|^p$, we have $F_p \geq 0$.

Estimates of $F_p(a,b) = |b|^p - |a|^p - pa^{<p-1>}(b-a)$

Write $f \approx g$ (resp. $f \lesssim g$), if there is a constant $c > 0$ such that $(1/c)f(x) \le g(x) \le cf(x)$ (resp. $f(x) \le cg(x)$) for all x. We have

$$F_p(a,b) \approx (b-a)^2(|a|+|b|)^{p-2}, \quad a,b \in \mathbb{R}.$$

In particular, $F_p(a,b) \approx (b-a)^2(|a|^{p-2}+|b|^{p-2})$ for $p \ge 2$.

$|b-a|^p \lesssim F_p(a,b)$ if $p \ge 2$, and $F_p(a,b) \lesssim |b-a|^p$ if $p \in (1,2)$.

In general $F_p(a,b) \neq F_p(b,a)$, but

$$\begin{aligned} H_p(a,b) &:= \tfrac{1}{2}(F_p(a,b)+F_p(b,a)) \\ &= \frac{p}{2}(b^{<p-1>} - a^{<p-1>})(b-a) \end{aligned}$$

Furthermore, $F_p(a,b) \approx H_p(a,b) \approx (a^{<p/2>} - b^{<p/2>})^2$.

Douglas identity in L^p

Recall that $p \in (1, \infty)$ and $(b^{<p-1>} - a^{<p-1>})(b - a) \geq 0$.

Theorem ([5])

If $u = P_D[g]$, then

$$\frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(y)^{<p-1>} - u(x)^{<p-1>})(u(y) - u(x))\, \nu(x, y)\, \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{2} \iint\limits_{D^c \times D^c} (g(w)^{<p-1>} - g(z)^{<p-1>})(g(w) - g(z))\gamma_D(w, z)\, \mathrm{d}w\mathrm{d}z.$$

Here we assume the finiteness of the right-hand side, $\nu(x, y)$ is the Lévy density, $\gamma_D(w, z)$ is the interaction kernel, same as for $p = 2$.

Sobolev–Bregman spaces

In fact, for $u\colon \mathbb{R}^d \to \mathbb{R}$ we define

$$\mathcal{E}_D^{(p)}[u] := \frac{1}{p} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} F_p(u(x),u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{p} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} H_p(u(x),u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(y)^{\langle p-1\rangle} - u(x)^{\langle p-1\rangle})(u(y)-u(x))\nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Similarly, for $g\colon D^c \to \mathbb{R}$,

$$\mathcal{H}_D^{(p)}[g] := \frac{1}{2} \iint\limits_{D^c\times D^c} (g(z)^{\langle p-1\rangle} - g(w)^{\langle p-1\rangle})(g(z)-g(w))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z.$$

We prove in [5] *trace and extension theorems* for the spaces

$$\mathcal{V}_D^p := \{u|\ \mathcal{E}_D^{(p)}[u] < \infty\}, \qquad \mathcal{X}_D^p := \{g|\ \mathcal{H}_D^{(p)}[g] < \infty\}.$$

Some insights: Nonlinear Hardy-Stein

Recall that $F_p(a,b) = |b|^p - |a|^p - pa^{<p-1>}(b-a)$, $a, b \in \mathbb{R}$.
Since u is L-harmonic,

$$
\begin{aligned}
& L|u|^p(y) = L|u|^p(y) - pu(y)^{\langle p-1\rangle} Lu(y) \\
& = \lim_{\epsilon\to 0+} \int_{|z-y|>\epsilon} (|u(z)|^p - |u(y)|^p - pu(y)^{\langle p-1\rangle}(u(z)-u(y)))\nu(y,z)\,\mathrm{d}z \\
& = \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y,z).
\end{aligned}
$$

To get Hardy-Stein identity we use the Dynkin formula for $|u(x)|^p$:

Lemma ([5]; for $\Delta^{\alpha/2}$ see [3])

If $u = P_D[g]$ and $x \in D$, then $\int_{D^c} |g(z)|^p \, P_D(x,z)\mathrm{d}z$ equals

$$
|u(x)|^p + \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y.
$$

Some more insights

Douglas identity is proved by Hardy-Stein, mysterious cancellations and the following

Lemma

Let X be a random variable with $\mathbb{E}|X| < \infty$. Then,

$$\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}|X|^p - |\mathbb{E}X|^p \geq 0,$$

and

$$\mathbb{E}F_p(a, X) = F_p(a, \mathbb{E}X) + \mathbb{E}F_p(\mathbb{E}X, X), \quad a \in \mathbb{R}.$$

Note that

$$\mathcal{E}_D^{(p)}[u] \approx \mathcal{E}_D(u^{<p/2>}, u^{<p/2>}),$$

however our nonlinear Douglas identity is an exact equality.

Comments: Test functions

Lemma

For every $p > 1$ we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_{\mathbb{R}^d} \subseteq \mathcal{V}^p_D$.

Indeed, for $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$|\phi(x+z) + \phi(x-z) - 2\phi(x)| \leq M(1 \wedge |z|^2), \quad x, z \in \mathbb{R}^d.$$

This is in harmony with (the estimates of) $F_p(a, b)$, the Lévy measure condition on $\nu(x, y)$ and the definition of $\mathcal{V}^p_D$.

Analogous spaces defined in terms of p-increments $|u(x) - u(y)|^p$ are often trivial for $p \in (1, 2)$.

Comments: Potential theory in terms of L^p spaces.

For $\phi \in C_c^\infty(D)$ we have

$$\mathcal{E}_D^{(p)}[\phi] = -\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} L\phi(x)\,\mathrm{d}x.$$

Davies [9, Chapter 2 and 3] and Bakry [2] give fundamental calculations with forms and powers. For the semigroups of local generators see Langer and Maz'ya [18] and Sobol and Vogt [25, Theorem 1.1]. Liskevich and Semenov [19] use the L^p setting to analyze perturbations of Markovian semigroups.

For nonlocal operators we refer to Farkas, Jacob and Schilling [11, (2.4)], and to Jacob [15, (4.294)]. One problem is to define $\mathcal{E}_D^{(p)}(u, v)$ and Schilling and Jacob [16] propose a choice. Hoh and Jacob study L^p variational theory and potential theory for submarkovian semigroups in [14]. Jacob and Schilling develop it further in [17].

Comments/Questions:

What is the semigroup? The question is about the (Servadei-Valdinoci) semigroup or Markov process corresponding to the quadratic form discussed at the beginning of the talk. We work on this. Related work: Zoran Vondraček [27].

Is $\mathcal{V}_D^p := \{u | \ \mathcal{E}_D^{(p)}[u] < \infty\}$ a good setting for direct variational methods for (nonlocal) PDEs?

We have some analogues for semigroups and martingales...

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