# The Korenblum Maximum Principle for Some Function Spaces

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# Outline

# Abstract

We study the Korenblum Maximum Principle for the Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$ and the weighetd Bergman space $A^p_\alpha(\mathbb{C})$ under exponential weights $e^{-\frac{p\alpha}{2}|z|^2}$.

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- Finally, we show a failure of the Korenblum Maximum Principle for weighted Bergman space.

This is a joint work with Wee JunJie (NTU, Singapore)

# 1. Basic definitions and notation

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathcal{O}(\mathbb{D})$ denotes the space of holomorphic functions on $\mathbb{D}$ endowed with the compact-open topology and $\mathcal{O}(\mathbb{C})$ denotes the space of entire functions on $\mathbb{C}$ endowed with the compact-open topology.

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## Weighted Hardy space

For $0 < p < \infty$, $\alpha \geq 0$, the weighted Hardy space $H^p_\alpha(\mathbb{D})$ consists of functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{H^p_\alpha} = \sup_{0 \leq r < 1} \left[ (1-r)^\alpha \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right] < \infty.$$

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- When $\alpha = 0$, we have the Hardy space $H^p(\mathbb{D})$.

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- When $\alpha = 0$, we have the Hardy space $H^p(\mathbb{D})$.
- When $p = \infty$, we have the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$, where $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|$.

## Weighted Bergman space

Let $0 < p < \infty$, $\alpha > -1$ and we consider the exponential weight $e^{-\frac{p\alpha}{2}|z|^2}$. The weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{A^p_\alpha} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\right]^{\frac{1}{p}} < \infty,$$

where $dA(z) = dx\, dy = r\, dr\, d\theta,\ z = x + iy = re^{i\theta}$, is the Lebesgue measure on $\mathbb{C}$.

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- If $p = \infty$, then $A^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$.
- For $0 < p < \infty$ and $\alpha = 0$, we have the standard Bergman space $A^p(\mathbb{D})$.

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When $p = 2$, we have the classical Bergman space $A^2(\mathbb{D})$, i.e. the space of functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{A^2} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \, dA(z)\right]^{\frac{1}{2}} < \infty.$$

## Weighted Fock space

For $0 < p < \infty$, $\alpha > 0$, the weighted Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$ consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$, for which

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- For the case $p = \infty$ and $\alpha > 0$, we have

$$\|f\|_{\mathcal{F}^\infty_\alpha} = \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

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It is well known that $\mathcal{F}^p(\mathbb{C})$ with $1 \leq p \leq \infty$ is a Banach space, while for $0 < p < 1$, $\mathcal{F}^p(\mathbb{C})$ is a complete metric space with distance $d(f, g) = ||f - g||^p_p$.

## 2. Korenblum's conjecture and development

In 1991, Boris Korenblum stated the following conjecture for $A^2(\mathbb{D})$.

**Conjecture 2.1**

*There exists a numerical constant c,* $0 < c < 1$*, such that if f and g are holomorphic in* $\mathbb{D}$ *and* $|f(z)| \leq |g(z)|$ $(c < |z| < 1)$*, then* $\|f\|_{A^2} \leq \|g\|_{A^2}$*.*

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We call $c$ a Korenblum constant and denote by $\kappa$ as the largest value of $c$. So far, the exact value of $\kappa$ is still not known.

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We call $c$ a Korenblum constant and denote by $\kappa$ as the largest value of $c$. So far, the exact value of $\kappa$ is still not known.

- Korenblum proved with a counterexample provided by Martin that $\kappa_{A^2} < \frac{1}{\sqrt{2}}$.

- A series of partial results were then produced along the way by Korenblum, O'Neil, Richards and Zhu , Korenblum and Richards, Matero, Schwick and others.

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- In 1999, the conjecture was first proven of its existence by Hayman with $c = 0.04$ for the space $A^2(\mathbb{D})$.

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- In the same year, Hinkkanen proved the existence of $c$ for $A^p(\mathbb{D})$, $1 \leq p < \infty$, with $c = 0.15724$.

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Thereafter, several works have been carried out in finding lower bounds and upper bounds of Korenblum constants.

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- In 2012, Zhu showed that it is possible to choose some $c$ for the Korenblum Maximum Principle to hold in $\mathcal{F}^p_\alpha(\mathbb{C})$ where $\alpha > 0$ and $p \geq 1$.

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- In 2019, Lou and Hu also disproved the Korenblum Maximum Principle for general Fock Space $\mathcal{F}^p_\alpha(\mathbb{C})$ where $0 < p < 1$, $\alpha > 0$.

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In order to avoid ambiguity, we denote the largest Korenblum constant, unless specified otherwise, by $\kappa_{A^p_\alpha}$ for $A^p_\alpha(\mathbb{D})$, $\kappa_{\mathcal{F}^p_\alpha}$ for $\mathcal{F}^p_\alpha(\mathbb{C})$, etc.

For a better illustration, we summarize the main developments of $H^p_\alpha(\mathbb{D})$, $A^p_\alpha(\mathbb{D})$ and $\mathcal{F}^p_\alpha(\mathbb{C})$ in Tables 1, 2 and 3, respectively.

| | $\alpha$ | $p$ | **Results** |
|---|---|---|---|
| $H^p_\alpha(\mathbb{D})$ | $\alpha = 0$ | $0 < p \leq \infty$ | 1998: For $E \subset \mathbb{D}$, $E$ is dominating if and only if $E$ is non-tangentially dense. |
| $\alpha \geq 0$ | $\alpha > 0$ | $0 < p \leq \infty$ | **No Development** |

Table 1: Main Development on $H^p_\alpha(\mathbb{D})$

# Table 2: Main Development on $A^p_\alpha(\mathbb{D})$

| | $\alpha$ | $p$ | Results | |
|---|---|---|---|---|
| $A^p_\alpha(\mathbb{D})$ $\alpha > -1$ | $\alpha = 0$ | $p = 2$ | 1999: $c \geq \frac{1}{25}$<br>2006: $c \geq 0.21$<br>2006: $c \geq 0.25018$<br>2011: $c \geq 0.28185$ | 1991: $c < \frac{1}{\sqrt{2}}$<br>2003: $c < 0.69472$<br>2004: $c < 0.685086$<br>2004: $c < 0.67794$ |
| | | $0 < p < 1$ | 2018: $c$ does not exist | |
| | | $1 \leq p < \infty$ | 1999: $c \geq 0.15724$<br>2006: $c \geq 0.1921$<br>2011: $c \geq 0.23917$ | **No Development on Upper Bound** |
| | | $p = \infty$ | $A^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$ (see $H^\infty(\mathbb{D})$) | |
| | $\alpha \neq 0$ | $0 < p \leq \infty$ | **No Development** | |

Table 2: Main Development on $A^p_\alpha(\mathbb{D})$

# Table 3: Main Development on $\mathcal{F}^p_\alpha(\mathbb{C})$

| | $\alpha$ | $p$ | **Results** | |
|---|---|---|---|---|
| $\mathcal{F}^p_\alpha(\mathbb{C})$ $\alpha > 0$ | $\alpha = 1$ | $p = 2$ | 2006: $c \geq 0.54$<br>2006: $c \geq 0.7248$ | **No Development on Upper Bound** |
| | | $0 < p < 1$ | 2018: $c$ does not exist | |
| | | $1 \leq p \leq \infty$ | $c$ **exists but no development on upper bound** | |
| | $\alpha \neq 1$ | $0 < p < 1$ | 2018: $c$ does not exist | |
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Table 3: Main Development on $\mathcal{F}^p_\alpha(\mathbb{C})$

# 3. Auxiliaries

We recall two results by Korenblum and Božin and Karapetrović respectively, which are closely related to our main results for $A^p_\alpha(\mathbb{D})$ in this talk.

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**Theorem 3.1 (Korenblum, 1991)**

*Let $c > \frac{1}{\sqrt{2}}$. There exist functions $f$ and $g$ in $A^2(\mathbb{D})$ such that $|f(z)| \leq |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^2} > \|g\|_{A^2}$. Therefore, $\kappa_{A^2} \leq \frac{1}{\sqrt{2}}$.*

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**Theorem 3.2 (Božin and Karapetrović, 2018)**

*Let $0 < p < 1$ and $0 < c < 1$. There exist functions $f$ and $g$ in $A^p(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$ and $\|f\|_{A^p} > \|g\|_{A^p}$.*

The following lemma is used to prove our main result.

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## Lemma 3.3

*Let* $0 < p < 1$, $\alpha > 0$ *and* $0 < c < 1$. *Then there exist positive integer* $n$ *and* $0 < \delta < 1$, *such that*

$$2\delta^{np+2}\left(\int_0^1 ue^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \int_1^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\right) > \left(1+\left(\frac{\delta}{c}\right)^n\right)^p\left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du.$$

# 4. Main results

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- **Fock Spaces $\mathcal{F}^p_\alpha(\mathbb{C})$**

## • Fock Spaces $\mathcal{F}^p_\alpha(\mathbb{C})$

We first consider the general Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$, where $p \geq 1$, $\alpha > 0$ and find an *upper bound* for $\kappa_{\mathcal{F}^p_\alpha}$.

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**Theorem 4.1 (Upper Bound of $\kappa_{\mathcal{F}^p_\alpha}$)**

*Let $p \geq 1$, $\alpha > 0$ and*

$$c > \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right)}.$$

*There exist functions $f$ and $g$ in $\mathcal{F}^p_\alpha(\mathbb{C})$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but $\|f\|^p_{\mathcal{F}^p_\alpha} > \|g\|^p_{\mathcal{F}^p_\alpha}$. Therefore,*

$$\kappa_{\mathcal{F}^p_\alpha} \leq \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right)}.$$

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## Corollary 4.2

*Let $c > 1$. There exist functions $f$ and $g$ in $\mathcal{F}^2(\mathbb{C})$ such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|^2_{\mathcal{F}^2} > \|g\|^2_{\mathcal{F}^2}$. Therefore, $\kappa_{\mathcal{F}^2} \leq 1$.*

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It is also interesting to note that when $p = 1$ and $\alpha = \frac{1}{2}$, we have

$$\kappa_{\mathcal{F}^1_{0.5}} \leq \sqrt{4} \cdot \Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}.$$

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## Remark 4.3

Theorem and Corollary above filled the gaps in Table 3.

- **Bergman Spaces $A^p_\alpha(\mathbb{D})$**

## • Bergman Spaces $A^p_\alpha(\mathbb{D})$

The results about the failures of Korenblum Maximum Principle in $A^p(\mathbb{D})$, $0 < p < 1$ (Božin and Karapetrović, 2018) and in $\mathcal{F}^p_\alpha(\mathbb{C})$, $0 < p < 1$, $\alpha > 0$ (Hu and Lou, 2019) inspire us to be interested in a question whether there is any failure of the Korenblum Maximum Principle for the weighted Bergman space $A^p_\alpha(\mathbb{D})$, $0 < p < 1$, $\alpha \neq 0$.

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It turns out that the failure exists and the result below not only proves this fact, but also generalizes Theorem 3.2 for any $\alpha > 0$.

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**Theorem 4.4**

*Let* $0 < p < 1$ *and* $\alpha > 0$. *Suppose* $0 < c < 1$. *Then there exist functions* $f$ *and* $g$ *in* $A^p_\alpha(\mathbb{D})$ *such that* $|f(z)| < |g(z)|$ *for any* $z$ *with* $c < |z| < 1$ *and* $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$.

Lastly, we have the upper bounds for the space $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$.

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### Theorem 4.5 (Upper Bound of $\kappa_{A^p_\alpha}$)

*Let $p \geq 1$, $\alpha \geq 0$. Consider the Bergman space $A^p_\alpha(\mathbb{D})$.*

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*Let $p \geq 1$, $\alpha \geq 0$. Consider the Bergman space $A^p_\alpha(\mathbb{D})$.*

1) *For $\alpha = 0$, suppose* $\left(\dfrac{2}{p+2}\right)^{\frac{1}{p}} < c < 1$.

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*There exist functions $f$ and $g$ in $A^p_\alpha(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$.*

## Remark 4.6

Clearly, in order to have the Korenblum Maximum Principle for $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$, we must have

$$\kappa_{A^p_\alpha} \leq \begin{cases} \left(\dfrac{2}{p+2}\right)^{\frac{1}{p}}, & \alpha = 0, \\ \sqrt[p]{\dfrac{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \displaystyle\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}} e^{-u} du}{(1 - e^{-\frac{p\alpha}{2}})}}, & \alpha > 0. \end{cases}$$

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## Remark 4.7

Theorems 4.4 and 4.5 have filled in some gaps in Table 2 with partial generalization.

## 5. Concluding remarks and open questions

In summary, we obtained explicit expressions for the upper bounds of $\mathcal{F}^p_\alpha(\mathbb{C})$ (Theorem 4.1) and $A^p_\alpha(\mathbb{D})$, where $p \geq 1$ and $\alpha \geq 0$ (Theorem 4.5). We also proved that the Korenblum Maximum Principle fails for $A^p_\alpha(\mathbb{D})$, $0 < p < 1$, $\alpha > 0$ (Theorem 4.4), thereby obtaining greater closure to the Korenblum Maximum Principle for the Bergman spaces.

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Theorems 4.1 and 4.5 led us to the following open questions.

**Question 5.1**

*Let $c = 1$. Do there exist functions $f(z)$ and $g(z)$ in $\mathcal{F}^2(\mathbb{C})$ for which $|f(z)| < |g(z)|$ with all $|z| > c$ and $\|f\|_{\mathcal{F}^2} > \|g\|_{\mathcal{F}^2}$?*

## Question 5.2

*Let* $p \geq 1$, $\alpha > 0$ *and*

$$c = \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right)}.$$

*Do there exist functions* $f(z)$ *and* $g(z)$ *in* $\mathcal{F}^p_\alpha(\mathbb{C})$ *for which* $|f(z)| < |g(z)|$ *with all* $|z| > c$ *and* $\|f\|_{\mathcal{F}^p_\alpha} > \|g\|_{\mathcal{F}^p_\alpha}$?

## Question 5.3

*Let* $p \geq 1$, $\alpha \geq 0$ *and*

$$c = \begin{cases} \left(\frac{2}{p+2}\right)^{\frac{1}{p}}, & \alpha = 0 \\ \sqrt[p]{\dfrac{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \displaystyle\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}} e^{-u} du}{(1 - e^{-\frac{p\alpha}{2}})}}, & \alpha > 0. \end{cases}$$

*Do there exist functions* $f(z)$ *and* $g(z)$ *in* $A^p_\alpha(\mathbb{D})$ *for which* $|f(z)| < |g(z)|$ *with all* $c < |z| < 1$ *and* $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$ *?*

## Remark 5.4

- Note that solving Question 5.3 indirectly generalizes the counterexample by Martin in [Korenblum, 1991] for $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$.

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- Note also that the case of $A^p_\alpha(\mathbb{D})$ where $-1 < \alpha < 0$ and $0 < p < \infty$ still remains unsolved.

**Thank you !**