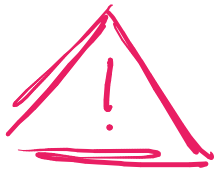
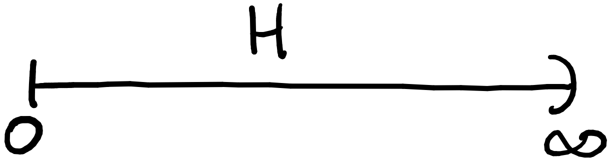
TOC



Sometimes I neglect exceptional cases without writing them explicitly.

Canonical systems on the half line



$H : [0,\infty) \to \mathbb{R}^{2\times 2}$

$H(t) = H(t)^*$ a.e.

$H(t) \neq 0$ a.e.

$\forall T > 0.\ H \in L^1(0,T)$

$H \notin L^1(0,\infty)$

$H(t) \geq 0$ a.e.

Hamiltonian

$$y'(t) = zJH(t)y(t)$$ canonical system

$\left(J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ z \in \mathbb{C}\right.$ eigenvalue parameter $\left.\right)$

The operator model

- Hilbert space $L^2(H)$: certain closed subspace of

$$\left\{ f : [0,\infty) \to \mathbb{C}^2 \;\middle|\; f \text{ measurable}, \int_0^\infty f^* H f < \infty \right\}$$

modulo " = a.e. "

- maximal operator $T_{max}(H)$: its graph is

$$\left\{ (f,g) \in L^2(H) \times L^2(H) \;\middle|\; \begin{array}{l} f \text{ has a.c. representant}, \\ f' = JHg \text{ a.e.} \end{array} \right\}$$

- boundary map $\Gamma(H)$:

$$\begin{cases} T_{max}(H) \longrightarrow \mathbb{C}^2 \\ (f,g) \longmapsto f(0) \end{cases}$$

$\rightsquigarrow$ selfadjoint model operator $A(H)$:

its graph is

$$\{(f,g) \in T_{max}(H) \mid (1,0)f(0) = 0\}$$

TASK : understand the spectrum of $A(H)$

The Weyl coefficient

The fundamental solution $\omega(t,z)$ is the unique solution of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t,z)J = z\,\omega(t,z)H(t), & t\in(0,\infty) \text{ a.e.} \\ \omega(0,z) = I \end{cases}$$

Action of $GL(2,\mathbb{C})$ on $\overline{\mathbb{C}}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * z = \frac{az+b}{cz+d}$$

For $t \in [0,\infty)$ and $z \in \mathbb{C}^+$ the Weyl disk $\Omega_{t,z}$ is

$$\Omega_{t,z} = \omega(t,z) * \overline{\mathbb{C}^+}$$

- $\forall t \in [0,\infty), z \in \mathbb{C}^+ .\ \Omega_{t,z} \subseteq \overline{\mathbb{C}^+}$
- $\forall t, s \in [0,\infty), z \in \mathbb{C}^+ .\ t \leq s \Rightarrow \Omega_{t,z} \supseteq \Omega_{s,z}$
- $\forall z \in \mathbb{C}^+ .\ \left| \bigcap_{t \in [0,\infty)} \Omega_{t,z} \right| = 1$

The Weyl coefficient q_H is the function on $\mathbb{C}^+$ with

$$\bigcap_{t \in [0,\infty)} \Omega_{t,z} = \{ q_H(z) \}$$

- q_H is a Nevanlinna function, i.e. analytic on $\mathbb{C}^+$ with $\operatorname{Im} q_H \geq 0$ (or $q_H = \infty$).

Herglotz integral representation of q_H:

$$q_H(z) = a_H + b_H z + \int_{-\infty}^{\infty} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu_H(x)$$

$$\left(a_H \in \mathbb{R},\ b_H \geq 0,\ \mu_H \text{ positive measure},\ \int_{-\infty}^{\infty} \frac{d\mu_H(x)}{1+x^2} < \infty \right)$$

Direct spectral theorem

For $f \in L^2(H)$, supp supp $f < \infty$:

$$(\mathcal{F}f)(x) = \int_0^\infty (0,1)\,\omega(t,z)\,H(t)\,f(t)\,dt$$

For $F \in L^2(\mu_H)$, supp F compact :

$$(\mathcal{G}F)(t) = \int_{-\infty}^\infty \left[(0,1)\,\omega(x,z)\right]^T F(x)\,d\mu_H(x)$$

Theorem:

$$\begin{tikzcd} L^2(H) \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\cong"] & L^2(\mu_H) \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

$$A(H) \rightsquigarrow \text{multiplication with } x$$

Inverse spectral theorem

Theorem:

(i) $\forall$ q Nevanlinna function $\exists$ H . $q = q_H$

(ii) H in (i) is unique (up to changes of scale)

Operator theory / complex analysis / canonical systems

- studying selfadjoint operators with simple spectrum

  $\cong$ studying canonical systems (with finite σ_H)

- studying the unit ball in $H^\infty(\mathbb{D})$

  $\cong$ studying canonical systems

Weyl coefficient vs. spectral measure

Spectral properties of $A(H)$ — i.e. μ_H — can be studied via q_H

Classical Abelian-Tauberian theorems tell that

behaviour of μ_H towards $\pm\infty$ $\Longleftrightarrow$ behaviour of q_H towards $+i\infty$

High-energy behaviour

$\boxed{A}$ μ_H is finite $\wedge\ b_H = 0 \iff \limsup\limits_{r\to\infty} r \operatorname{Im} q_H(ir) < \infty$

$\boxed{B}$ $\mu_H((-x,x)) = \delta x^\gamma (1+o(1)) \ \wedge\ b_H = 0$

$\wedge \ \lim\limits_{x\to\infty} \dfrac{\mu_H((0,x))}{\mu_H((-x,0))} \in [0,\infty]$ exists

$\iff q_H(ir) = i\omega \cdot r^{\gamma-1} \cdot (1+o(1)),\ r\to\infty$

$\boxed{C}$ $\displaystyle\int_{-\infty}^{\infty} \frac{d\mu_H(x)}{1+|x|^\gamma} < \infty \ \wedge\ b_H = 0 \iff \int_1^\infty \frac{\operatorname{Im} q_H(ir)}{r^\gamma}\, dr < \infty$

$(0<\gamma<2)$

(Instead of powers r^γ or x^γ one can use functions which are regularly varying in Karamata sense)

TASK

$$\begin{tikzcd}
\text{properties of } H \arrow[r, bend left, "\text{direct spectral relations}"] & \text{properties of } \rho_H \arrow[l, bend left, "\text{inverse spectral relations}"]
\end{tikzcd}$$

Theorem A:

[Winkler 2000] probably earlier

$$\limsup_{r\to\infty} r \operatorname{Im} q_H(ir) < \infty \iff$$

$$\exists\, c>0,\ \varphi\in(0,\pi):\ \ker H(t) = \operatorname{span}\left\{\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\right\},\quad t\in(0,c) \text{ a.e.}$$

Notice:

$$\exists\, c>0:\ \ker H(t) = \operatorname{span}\left\{\begin{pmatrix}1\\ 0\end{pmatrix}\right\},\ t\in(0,c) \text{ a.e.}$$

$$\iff b_H > 0$$

A measurable function $f:(0,\infty)\to(0,\infty)$ is regularly varying at ∞ with index $\varrho\in\mathbb{R}$, if

$$\forall \lambda>0.\ \lim_{r\to\infty}\frac{f(\lambda r)}{f(r)}=\lambda^{\varrho}$$

(analogous at 0, for $\varrho=\pm\infty$ rapidly varying)

Theorem B: [Langer, Pruckner, Woracek unpublished]

Write $H=\begin{pmatrix} h_1 & h_3 \\ h_3 & h_2\end{pmatrix}$, $m_j(t)=\int_0^t h_j$, $\mathfrak{f}:=m_1+m_2$.

Assume neither h_1 nor h_2 vanishes on any neighbourhood of 0, and $\mathfrak{f}$ is regularly varying at 0 with positive index.

Let $\mathring{t}:(0,\infty)\to(0,\infty)$ be the function with

$$\forall r>0.\ (m_1 m_2)(\mathring{t}(r))=\frac{1}{r^2}$$

(i) $\exists\ \varpi$ regularly varying at ∞, $\omega \in \mathbb{C}\setminus\{0\}$.

$$q_H(ir) = i\omega\, \varpi(r)\,(1+o(1)), \quad r \to \infty$$

$\Updownarrow$

(ii) m_1, m_2 regularly or rapidly varying at 0, and

$$\delta := \lim_{t\to 0} \frac{m_3(t)}{\sqrt{m_1(t)\,m_2(t)}} \text{ exists.}$$

$\Updownarrow$

(iii)
$$q_H(rz) = i\omega \left(\frac{z}{i}\right)^{\alpha} \left(\frac{m_1(\mathring{t}(r))}{m_2(\mathring{t}(r))}\right)^{1/2} (1+o(1)), \quad r \to \infty,$$

where $o(1)$ is locally uniformly for $z \in \mathbb{C}^+$.

α is the index of $\left(m_1(\mathring{t}(r))/m_2(\mathring{t}(r))\right)^{1/2}$

ω can be computed from α and δ.

Theorem C:

[Langer, Pruckner, Woracek 2021 arXiv]

Assume neither h_1 nor h_2 vanishes on any neighbourhood of 0.

Let f be regularly varying at ∞, nondecreasing, continuous.

(i) $\exists c>0.\ \int_0^c h_1(t) \cdot f\left((m_1 m_2)(t)^{-\frac{1}{2}}\right) dt < \infty$

$\Downarrow$

(ii) $\int_1^\infty \ln q_H(ir) \frac{d f(r)}{r} < \infty$

$\Downarrow$

(iii) $\exists c>0.$

$$\int_0^c \begin{pmatrix} 1 \\ -m_3(t)/m_2(t) \end{pmatrix}^* H(t) \begin{pmatrix} 1 \\ -m_3(t)/m_2(t) \end{pmatrix} \cdot f\left((m_1 m_2)(t)^{-\frac{1}{2}}\right) dt < \infty$$

For $f : \mathbb{D} \to \mathbb{D}$ analytic, $\zeta \in S^1$,

$$\mathcal{C}_\zeta(f) := \{ w \in \overline{\mathbb{D}} \mid \exists\, r_n \uparrow 1 .\ f(r_n \zeta) \to w \} .$$

Theorem: [Belna, Colwell, Piranian 1985]

For $n \in \mathbb{N}$ let $\zeta_n \in S^1$, $K_n \subseteq \overline{\mathbb{D}}$ nonempty closed connected.

$\exists$ Blaschke product f $\forall n \in \mathbb{N} .\ \mathcal{C}_f(\zeta_n) = K_n$.

Theorem: [Pruckner, Woracek 2021]

Let $K \subseteq \overline{\mathbb{C}^+}$ nonempty closed connected.

$\exists H .\ \mathcal{C}_{q_H}(i\infty) = K$ and H is constructed explicitly.

The outer angular cluster set at ζ is

$$\mathcal{C}_{\measuredangle,\zeta}(f) = \{ \omega \in \overline{\mathbb{D}} \mid \exists z_n \xrightarrow{\measuredangle} \zeta \,.\, f(z_n) \to \omega \}$$

Every set $\mathcal{C}_{\measuredangle,\zeta}(f)$ is an increasingly countable union of nonempty closed connected sets.

Conjecture ? :   [Gauthier 2021]

Given $K \subseteq \overline{\mathbb{D}}$ which is an increasing countable union of nonempty closed connected sets. Then there exists $f : \mathbb{D} \to \mathbb{D}$ analytic with $\mathcal{C}_{\measuredangle,f}(1) = K$.

TASK : Given $K \subseteq \overline{\mathbb{C}^+}$ increasingly countable union of nonempty closed connected sets. Construct a Hamiltonian H with $\mathcal{C}_{\measuredangle,q_H}(i\infty) = K$.