Interpolation Problems for Vector-Valued de Branges-Rovnyak Spaces and Applications

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Outline

Part 1: Interpolation problems for Schur-class operator-valued functions

Part 2: Interpolation problems for functions in vector-valued de Branges-Rovnyak spaces

Part 3: Applications

Part 1: Interpolation problems for Schur-class operator-valued functions

The Schur class

$\mathcal{U}$, $\mathcal{Y}$, $\mathcal{X}$ = Hilbert spaces

$\mathcal{S}(\mathcal{U}, \mathcal{Y})$ = holomorphic functions S on $\mathbb{D}$ with values equal to contraction operators in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$

TFAE:

- $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$
- The de Branges-Rovnyak kernel $K_S(z, \zeta) := \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\overline{\zeta}}$ is a positive kernel on $\mathbb{D}$: $z_1, \dots, z_N \in \mathbb{D}$, $y_1, \dots, y_N \in \mathcal{Y}$, N=1,2,... $\Rightarrow \sum_{i,j=1}^{N} \langle K_S(z_i, z_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0$
- K has a Kolmogorov decomposition: $\exists\ H \colon \mathbb{D} \underset{\text{holo}}{\to} \mathcal{L}(\mathcal{X}, \mathcal{Y})$ s.t. $K(z, \zeta) = H(z)H(\zeta)^*$

The Schur class continued

$S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ also equivalent to:

- Unitary state-space realization: $\exists$ unitary system matrix $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ s.t. $S(z) = D + zC(I_{\mathcal{H}} - zA)^{-1}B$

Control motivation: Linear i/s/o linear system associated with $\mathbf{U}$:

$$\Sigma_{\mathbf{U}}: \begin{cases} x(n+1) = Ax(n) + Bu(n), & x(0) = x_0, \\ y(n) = Cx(n) + Du(n) \end{cases}$$

$n \in \mathbb{Z}_+$ = point in discrete time; above = "time-domain" equations

Control motivation continued

Application of *Z*-transform $\{w(n)\}_{n\geq 0} \mapsto \widehat{w}(z) := \sum_{n=0}^{\infty} w_n z^n$ converts "time-domain" equations

$$\Sigma_{\mathbf{U}}\colon \begin{cases} x(n+1) &= Ax(n) + Bu(n), \quad x(0) = x_0, \\ y(n) &= Cx(n) + Du(n) \end{cases}$$

to "freqeuncy-domain" equations

$$\widehat{\Sigma}_{\mathbf{U}}\colon \begin{cases} \widehat{x}(z) &= (I - zA)^{-1}x_0 + z(I - zA)^{-1}B\widehat{u}(z) \\ \widehat{y}(z) &= \mathcal{O}_{C,A}(z)x_0 + \Theta_{\mathbf{U}}(z)\widehat{u}(z) \end{cases}$$

where

- $\mathcal{O}_{C,A}(z) = C(I_{\mathcal{X}} - zA)^{-1}$ = the observabiliy operator of the system $\Sigma_{\mathbf{U}}$, and
- $\Theta_{\mathbf{U}}(z) = D + zC(I - zA)^{-1}B$ = the transfer function of the system $\Sigma_{\mathbf{U}}$

Special cases:

- $\mathbf{u} = 0 \Rightarrow \widehat{y}(z) = \mathcal{O}_{C,A}(z)x_0$ & $x_0 = 0 \Rightarrow \widehat{y}(z) = \Theta_{\mathbf{U}}(z)\widehat{u}(z)$

Control motivation continued II

Recall "frequency-domain" equations:

$$\widehat{\Sigma}_{\mathbf{U}}: \left\{ \begin{array}{lcl} \widehat{x}(z) & = & (I - zA)^{-1} x_0 + z(I - zA)^{-1} B \widehat{u}(z) \\ \widehat{y}(z) & = & \mathcal{O}_{C,A}(z) x_0 + \Theta_{\mathbf{U}}(z) \widehat{u}(z) \end{array} \right.$$

where

- $\mathcal{O}_{C,A}(z) = C(I_{\mathcal{X}} - zA)^{-1}$ = the observabiliy operator of the system $\Sigma \mathbf{U}$, and
- $\Theta_{\mathbf{U}}(z) = D + zC(I - zA)^{-1} B$ = the transfer function of the system $\Sigma_{\mathbf{U}}$

Furthermore, if $\mathbf{U}$ is unitary and A is stable $(A^n x_0 \underset{n \to \infty}{\to} 0$ in norm for each $x_0 \in \mathcal{X})$, then $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ is isometric, Θ is inner (i.e., $M_\Theta \colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ is isometric) and $\begin{bmatrix} \mathcal{O}_{C,A} & M_{\Theta_{\mathbf{U}}} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{U}} \end{bmatrix} \to H^2_{\mathcal{Y}}$ is unitary

(so in particular $H^2_{\mathcal{Y}} = \overline{\operatorname{Ran}}\, \mathcal{O}_{C,A} \bigoplus M_{\Theta_{\mathbf{U}}} H^2_{\mathcal{U}}$)

Alternative formulas for $\mathcal{O}_{C,A}(z)$ and $\Theta_{\mathbf{U}}(z)$

Slick formulas at the system-matrix level for $\mathcal{O}_{C,A}$ and $\Theta_{\mathbf{U}}(z)$:

- $\mathcal{O}_{C,A}(z) = \begin{bmatrix} 0 & I_{\mathcal{Y}} \end{bmatrix} \mathbf{U}(I_{\mathcal{X}\oplus\mathcal{U}} - zP_{\mathcal{X}\oplus\{0\}}\mathbf{U})^{-1} \begin{bmatrix} I_{\mathcal{X}} \\ 0 \end{bmatrix}$,
- $\Theta_{\mathbf{U}}(z) = \begin{bmatrix} 0 & I_{\mathcal{Y}} \end{bmatrix} \mathbf{U}(I_{\mathcal{X}\oplus\mathcal{U}} - zP_{\mathcal{X}\oplus 0}\mathbf{U})^{-1} \left[\begin{smallmatrix} 0 \\ I_{\mathcal{U}} \end{smallmatrix}\right]$

Thus $\mathbf{U}$ unitary and A stable $\Rightarrow$
$\begin{bmatrix} \mathcal{O}_{C,A} & M_{\Theta} \end{bmatrix} = M_{[0 \ I_{\mathcal{Y}}]U(I-zP_{\mathcal{X}\oplus 0}\mathbf{U})^{-1}} \colon \left[\begin{smallmatrix} \mathcal{X} \\ H^2_{\mathcal{U}} \end{smallmatrix}\right] \to H^2_{\mathcal{Y}}$ is unitary

Interpolation problem for Schur-class functions

Left-tangential Nevanlinna-Pick interpolation problem (LTNP)

Given points $z_1, \dots, z_N \in \mathbb{D}$ and vectors $a_1, \dots, a_N \in \mathcal{Y}$ and $c_1, \dots, c_N \in \mathcal{U}$ find $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ s.t. $a_i^* S(z_i) = c_i^*$ for $i = 1, \dots, N$

Motivation: H^∞-control (1980s-1990s)

LTOA point-evaluation and observability operators

Assume $(E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})$ is a output-stable pair:
$\mathcal{O}_{E,T} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ so $E(I - zT)^{-1}x = \sum_{n=0}^{\infty} ET^n x\, z^n \in H^2_{\mathcal{Y}}$ $\forall$
$x \in \mathcal{X}$

Define left-tangential operator-argument point-evaluation
$S \in H^\infty(\mathcal{U}, \mathcal{Y}) \mapsto (E^* S)^{\wedge L}(T^*) := \sum_{n=0}^{\infty} T^{*n} E^* S_n$ if
$S(z) = \sum_{n=0}^{\infty} S_n z^n$

Compute for $u \in \mathcal{U}$:
$\langle \sum_{n=0}^{\infty} T^{*n} E^* S_n u, x \rangle_{\mathcal{X}} = \sum_{n=0}^{\infty} \langle S_n u, ET^n x \rangle_{\mathcal{Y}} = \langle M_S u, \mathcal{O}_{E,T} x \rangle_{H^2_{\mathcal{Y}}}$
Note: (E, T) output-stable & $S \in H^\infty(\mathcal{U}, \mathcal{Y})$ $\Rightarrow$ series converges

Conclude $(E^* S)^{\wedge L}(T^*) = \mathcal{O}_{E,T}^* M_S|_{\mathcal{U}}$

LTNP vs LTOA interpolation

Example: $E^* = \begin{bmatrix} a_1^* \\ \vdots \\ a_N^* \end{bmatrix}$, $N^* = \begin{bmatrix} c_1^* \\ \vdots \\ c_N^* \end{bmatrix}$, $T^* = \begin{bmatrix} \overline{z}_1 & & \\ & \ddots & \\ & & \overline{z}_N \end{bmatrix}$

$$\Rightarrow (E^*S)^{\wedge L}(T^*) = \sum_{n=0}^{\infty} \begin{bmatrix} z_1^n & & \\ & \ddots & \\ & & z_N^n \end{bmatrix} \begin{bmatrix} a_1^* \\ \vdots \\ a_N^* \end{bmatrix} S_n = \begin{bmatrix} a_1^* S(z_1) \\ \vdots \\ a_N^* S(z_N) \end{bmatrix}$$

This equal to $N^* = \begin{bmatrix} c_1^* \\ \vdots \\ c_N^* \end{bmatrix}$ means $a_i^* S(z_i) = c_i^*$ for $i = 1, \dots, N$,

i.e.

Conclusion: LTOA point-evaluation interpolation

$(E^*S)^{\wedge L}(T^*) = N^*$ or $\mathcal{O}_{E,T}^* M_S|_{\mathcal{U}} = N^*$

for this example of (T, E, N) equivalent to

LTNP interpolation conditions $a_i^* S(z_i) = c_i^*$ for $i = 1, \dots, N$

Additional information on LTOA data set $\mathcal{D} = (T, E, N)$

Suppose

- $(E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})$ output-stable,
- $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$,
- $(E^* S)^{\wedge L}(T^*) = N^* \in \mathcal{L}(\mathcal{U}, \mathcal{X})$

Then

- (N, T) also output-stable and $\mathcal{O}_{E,T}^* M_S = \mathcal{O}_{N,T}^* \in \mathcal{L}(H^2_{\mathcal{U}}, \mathcal{X})$ $=$ extension of $\mathcal{O}_{E,T}^* M_S|_{\mathcal{U}} = N^* \in \mathcal{L}(\mathcal{U}, \mathcal{X}))$

Thus view LTOA interpolation as an equation in $\mathcal{L}(H^2_{\mathcal{U}}, \mathcal{X})$:
$\mathcal{O}_{E,T}^* M_S = \mathcal{O}_{N,T}^*$,

Positivity condition for solvability of LTOA(T,E,N)

Suppose LTOA(T, E, N) interpolaton problem has a solution, now written as $\mathcal{O}_{E,T}^* M_S = \mathcal{O}_{E,N}^*$ for some $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

Then: $\mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{E,T}^* M_S M_S^* \mathcal{O}_{E,T}$

$= \mathcal{O}_{E,T}^* \big(I_{H^2_{\mathcal{Y}}} - M_S M_S^* \big) \mathcal{O}_{E,T} \succeq 0$ since $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

$\Rightarrow P := \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} \succeq 0$ is a necessary condition for existence of solutions to LTOA int-problem

Deeper fact: $P \succeq 0$ also sufficient for existence of solutions to LTOAint-problem

Parametrization of solutions to LTOA int-problem

Special case: Assume T is strongly stable $(T^n x \underset{n\to\infty}{\to} \infty$ for $x \in \mathcal{X})$ and $P \succ 0$. Set $J = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix}$

Then there is an explicitly constructible (possibly unbounded) J-inner function $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$

(so $\Theta(z)^* J \Theta(z) = J$, $\Theta(z) J \Theta(z)^* = J$ for a.e. $z \in \mathbb{T}$

$M_\Theta\big|_{\text{dom}(M_\Theta)} = J$-unitary on $L^{2,J}_{\mathcal{Y}\oplus\mathcal{U}})$ so that:

S solves $\text{LTOA}(T, E, N) \Leftrightarrow \exists\, \mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ s.t.

$S(z) = \big(\Theta_{11}(z) + \Theta_{12}(z)\mathcal{E}(z)\big)\big(\Theta_{21}(z) + \Theta_{22}(z)\mathcal{E}(z)\big)^{-1}$

$=: T_{\Theta(z)}[\mathcal{E}(z)]$ (Chain-matrix linear-fractional transformation)

Construction of Θ

The algorithm starting with the data (T, E, N):
Set $C = \begin{bmatrix} E \\ N \end{bmatrix}$

- Construct a system matrix of the form $\mathbf{U} = \begin{bmatrix} T & B \\ C & D \end{bmatrix}$ (already have T and $C = \begin{bmatrix} E \\ N \end{bmatrix}$, must still solve for B, D so that $\mathbf{U} \begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix} \mathbf{U}^* = \begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix}$, $\mathbf{U}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathbf{U} = \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}$

  This comes down to finding

  $B \colon \mathcal{Y} \oplus \mathcal{U} \to \mathcal{X}$ and $D \colon \mathcal{Y} \oplus \mathcal{U} \to \mathcal{Y} \oplus \mathcal{U}$ solving the Cholesky factorization problem:

  $\begin{bmatrix} B \\ D \end{bmatrix} J \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix}$
- Then let $\Theta(z) = \Theta_{\mathbf{U}}(z)$ be the transfer function of the system $\Sigma_{\mathbf{U}}$: $\Theta(z) = D + zC(I - zT)^{-1}B$

Additional ingredients of the proof

Then also

- $\mathcal{O}_{E\oplus N,T}$ is isometric from $(\mathcal{X}^P)$ into $H^{2,J}_{\mathcal{Y}\oplus\mathcal{U}}$
- M_Θ is (possibly unbounded) J-unitary operator on $L^{2,J}_{\mathcal{Y}\oplus\mathcal{U}}$
- $\left(M_\Theta \cdot \{\text{polynomials in } H^{2,J}_{\mathcal{Y}\oplus\mathcal{U}}\}\right)^- = \text{Ran}\, \mathcal{O}^{\perp_J}_{E\oplus N,T}$

Then one can arrive at the statement S solves LTOA int-problem $\Leftrightarrow$ $S = T_\Theta(\mathcal{E})$ for some $\mathcal{E} \in \mathcal{S}(\mathcal{U},\mathcal{Y})$ (via either Ball-Helton Grassmannian approach or Potapov/Dym/Bolotnikov kernel-function approach) in a straightforward way

T not strongly stable

Without the strong stability assumption:

$\operatorname{Ran} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T} \underset{isom}{=} \mathcal{H}\left(\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}(z) P^{-1} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}(\zeta)^{*}\right) \underset{contr}{\subset} H^{2,J}_{\mathcal{Y} \oplus N}$

Θ not J -inner

$H^{2,J}_{\mathcal{Y} \oplus \mathcal{U}} = \operatorname{Ran} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T} + \left(\Theta \cdot (\text{polynomials})\right)^{-}$ is a Brangesian J-minimal decomposition and not a J-orthogonal decomposition

$\Rightarrow$ not clear how to proceed

$\Rightarrow$ motivation for a more flexible reformulation of the LTOA int-problem (Potapov operator-theory school Kharkiv, Ukraine)

LTOA int-problem reformulated: Preliminaries

Douglas lemma: Given $A \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_3)$, $B \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_3)$ $\exists$ $X \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ s.t. $\|X\| \le 1$ and $AX = B$
$\Leftrightarrow BB^* \preceq AA^* \Leftrightarrow \begin{bmatrix} I_{\mathcal{X}_2} & B^* \\ B & AA^* \end{bmatrix} \succeq 0$

Variant of Douglas lemma: Given $A \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_3)$, $B \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_3)$, $X \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$, then $\|X\| \le 1$ and $AX = B$ $\Leftrightarrow$

$$M := \begin{bmatrix} I_{\mathcal{X}_1} & B^* & X^* \\ B & AA^* & A \\ X & A^* & I_{\mathcal{X}_2} \end{bmatrix} \succeq 0 \text{ on } \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_3 \\ \mathcal{X}_2 \end{bmatrix}$$

Proof: Note by Schur-complement analysis $M \succeq 0 \Leftrightarrow$
$\begin{bmatrix} I_{\mathcal{X}_1} & B^* \\ B & AA^* \end{bmatrix} - \begin{bmatrix} X^* \\ A \end{bmatrix} \begin{bmatrix} X & A^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}_1} - X^*X & B^* - X^*A^* \\ B - AX & 0 \end{bmatrix} \succeq 0 \Leftrightarrow$
$\|X\| \le 1$ and $B = AX$

ASIDE: Thus original Douglas lemma is a matrix-completion problem: Given A, B, find X so that $M \succeq 0$
Many papers on this from the 1980s

Preliminaries: de Branges-Rovnyak spaces

Given a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$:

- The associated de Branges-Rovnyak kernel is
  $K_S(z, \zeta) = \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\overline{\zeta}}$
  with associated de Branges-Rovnyak space $= \mathcal{H}(K_S)$
  (RKHS with reproducing kernel K_S)
- In operator-theory form $\mathcal{H}(K_S) \underset{\text{isometrically}}{=} \operatorname{Ran}(I - M_S M_S^*)^{\frac{1}{2}}$
  with lifted norm, where $M_S \in \mathcal{L}(H^2_{\mathcal{U}}, H^2_{\mathcal{Y}})$ is the
  multiplication operator $M_S \colon f(z) \mapsto S(z)f(z)$

A positive-kernel reformulation of the LTOA int-problem

Given an admissible LTOA int-problem data set (T, E, N) (so (E, T) output-stable), and given $S \in \mathrm{Hol}_{\mathbb{D}}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$, set $F^S = \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} \in \mathcal{L}(X, H^2_{\mathcal{Y}})$, TFAE:

1. S solves the LTOA int-problem with data set $\mathcal{D} = (T, E, N)$
2. $\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix}$ satisfies $\mathbf{P} \succeq 0$
3. $\mathbf{K}(z, \zeta) = \begin{bmatrix} P & (I - \bar{\zeta} T^*)^{-1}(E^* - N^* S(\zeta)^*) \\ (E - S(z)N)(I - zT)^{-1} & \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\bar{\zeta}} \end{bmatrix}$ is a positive kernel
4. $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, $F^S x \in \mathcal{H}(K_S)$ with $\|F^S x\|_{\mathcal{H}(K_S)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}}$ $\forall x \in \mathcal{X}$
5. $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, $F^S x \in \mathcal{H}(K_S)$ with $\|F^S x\|_{\mathcal{H}(K_S)} = \|P^{\frac{1}{2}} x\|$ $\forall x \in \mathcal{X}$

$(2) \Leftrightarrow (3)$

$(2) \Leftrightarrow (3)$
Recall:

(2) $\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix}$ satisfies $\mathbf{P} \succeq 0$

(3) $\mathbf{K}(z,\zeta) = \begin{bmatrix} P & (I - \overline{\zeta} T^*)^{-1}(E^* - N^* S(\zeta)^*) \\ (E - S(z)N)(I - zT)^{-1} & \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\overline{\zeta}} \end{bmatrix}$ is a positive kernel

Proof: Note that
$\langle \mathbf{P} f, f \rangle_{\mathcal{X} \oplus H^2_{\mathcal{Y}}} = \sum_{j,\ell=1}^r \left\langle \mathbf{K}(z_j, z_\ell) \left[\begin{smallmatrix} x_\ell \\ y_\ell \end{smallmatrix} \right], \left[\begin{smallmatrix} x_j \\ y_j \end{smallmatrix} \right] \right\rangle_{\mathcal{X} \oplus \mathcal{Y}}$
where $f \in \mathcal{X} \oplus H^2_{\mathcal{Y}}$ is of the form $f = \sum_{j=1}^r \begin{bmatrix} x_j \\ k_{\mathrm{Sz}}(\cdot, z_j) y_j \end{bmatrix}$

$(1) \Rightarrow (5)$

$(1) \Rightarrow (5)$
Recall

(1) S solves the LTOA int-problem with data set $\mathcal{D} = (T, E, N)$

(5) $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, $F^S x \in \mathcal{H}(K_S)$ and $\|F^S x\|_{\mathcal{H}(K_S)} = \|P^{\frac{1}{2}}$

Note that

$$F^S = \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} = \mathcal{O}_{E,T} - M_S M_S^* \mathcal{O}_{E,T} = (I - M_S M_S^*) \mathcal{O}_{E,T}$$

$$\Rightarrow \|F^S x\|^2_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*) \mathcal{O}_{E,T} x, \mathcal{O}_{E,T} x \rangle_{H^2_{\mathcal{Y}}}$$

$$= \langle \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}) x, x \rangle_{\mathcal{X}} = \langle P x, x \rangle_{\mathcal{X}} = \|P^{\frac{1}{2}} x\|^2_{\mathcal{X}}$$

$(4) \Leftrightarrow (2)$

$(4) \Leftrightarrow (2)$
Recall:

(2) $\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix}$ satisfies $\mathbf{P} \succeq 0$

(4) $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y}), \quad F^S x \in \mathcal{H}(K_S)$ with $\|F^S x\|_{\mathcal{H}(K_S)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}}$ $\forall$ $x \in \mathcal{X}$

Proof: Slightly finer Schur-complement argument

(2) ⇔ (1):

Recall:

(1) S solves the LTOA int-problem with data set $\mathcal{D} = (T, E, N)$

(2) $\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}} \end{bmatrix}$ satisfies $\mathbf{P} \succeq 0$

Proof:

Suppose $\mathbf{P} \succeq 0 \;\Rightarrow I - M_S M_S^* \succeq 0$, i.e., $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

From the definitions $\mathbf{P} = \begin{bmatrix} \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} & \mathcal{O}_{ET}^* - \mathcal{O}_{N,T}^* M_S^* \\ \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} & I - M_S M_S^* \end{bmatrix} \succeq 0$

By a Schur-complement argument

$$\Leftrightarrow \widehat{\mathbf{P}} := \begin{bmatrix} I_{H^2_{\mathcal{U}}} & \mathcal{O}_{N,T} & M_S^* \\ \mathcal{O}_{N,T}^* & \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} & \mathcal{O}_{E,T}^* \\ M_S & \mathcal{O}_{E,T} & I_{H^2_{\mathcal{Y}}} \end{bmatrix} \succeq 0$$

Now Douglas-lemma variant $\Rightarrow \|M_S\| \leq 1$ (as already known) and $\mathcal{O}_{N,T} = M_S^* \mathcal{O}_{E,T}$, i.e.,

S solves LTOAint-problem and (2) ⇒ (1).

(1) ⇒ (2): The steps are reversible.

Conclusions

Note: Reliance on Krein-space geometry (difficult to interpret when strong stability assumption is not present) is eliminated; Instead all the analysis is manipulation of positive kernels

Conclusions 2

Formulation of LTOA(T,E,N) int-problem appears to require that $\mathcal{O}_{E,T}$ and $\mathcal{O}_{N,T}$ be bounded (in $\mathcal{L}(\mathcal{X}, H^2_{\mathcal{Y}})$ and $\mathcal{L}(\mathcal{X}, H^2_{\mathcal{U}})$ respectively)

However (2),(3),(4) in positive-kernel reformulation theorem make sense if

- we take P equal to any positive-semidefinite operator on $\mathcal{X}$, and
- Assume that $\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],T} \colon x \mapsto \left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right](I - zT)^{-1}$ maps $\mathcal{X}$ into $\text{Hol}_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{D})$ (holomorphic functions on $\mathbb{D}$ with values in $\mathcal{Y} \oplus \mathcal{U}$)

Furthermore, we still have $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ if we also assume $P \succeq 0$ solves $P - T^*PT = C^*JC$, where $C = \left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right]$

(If T strongly stable, $P = \mathcal{O}^*_{E,T}\mathcal{O}_{E,T} - \mathcal{O}^*_{N,T}\mathcal{O}_{N,T}$ is the unique solution)

The aAIP

This suggests: Assume that (T, E, N, P) is admissible data set for aAIP:

- $\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T} \colon \mathcal{X} \to \mathrm{Hol}_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{D}$
- $P \succeq 0$ satisfies $P - T^* P T = C^* J C$, where $C = \left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right]$

Then we can take any of (2), (3), (4) as the definition of a more general problem: we shall take (4) as the Definition.

The analytic Abstract Interpolation Problem

Analytic Abstract Interpolation Problem aAIP(T, E, N, P)

Given $\mathcal{D} = (T, E, N, P)$ with $T \in \mathcal{L}(\mathcal{X})$, $\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right] \in \mathcal{L}(\mathcal{X}, \mathcal{Y} \oplus \mathcal{U})$, $\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T} \colon \mathcal{X} \to \mathrm{Hol}_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{D})$, find all $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ so that

(4) $F^S := \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} \colon \mathcal{X} \to \mathcal{H}(K_S)$ with $\|F^S x\| \le \|P^{\frac{1}{2}} x\|$

Theorem on solution of aAIP(T, E, N, P):

Given aAIP admissible data set T, E, N, P, TFAE:

(4) S is a solution of the aAIP(E, N, T, P)

(2) $\mathbf{P} = \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} \succeq 0$

(3) $\mathbf{K}(z, \zeta) = \begin{bmatrix} P & (I - \overline{\zeta} T^*)^{-1}(E^* - N^* S(\zeta)^*) \\ (E - S(z)N)(I - zT)^{-1} & \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\overline{\zeta}} \end{bmatrix}$ is a positive kernel

LFT parametrization of solution set

Furthermore, if $P \succ 0$ and if Θ is constructed as above, then any solution S has the form
$S(z) = (\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z))(\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z))^{-1}$ for $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

Smooth proof starting with (4) instead of old (1): By formulation (4) of a solution (now the definition of a solution), S solves $\Leftrightarrow$
$(*)$ $F^S := \begin{bmatrix} I & -M_S \end{bmatrix} \begin{bmatrix} \mathcal{O}_{E,T} \\ \mathcal{O}_{N,T} \end{bmatrix}$ maps $\mathcal{X}^P$ contractively into $\mathcal{H}(K_S)$.
But by general RKHS results,
$\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T} \colon \mathcal{X}^P \underset{\text{isom.}}{\to} \mathcal{H}(K^P_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}) = \mathcal{H}(K_\Theta^{J,J})$.
$K^P_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}(z, \zeta) := \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}(z) P^{-1} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], T}(\zeta)^*$ while
$K_\Theta^{J,J}(z, \zeta) = \frac{J - \Theta(a) J \Theta(\zeta)^*}{1 - z\overline{\zeta}}$
Now use a (not hard) general result that says property $(*)$ characterizes $S \in \operatorname{Ran} T_\Theta$ $\Rightarrow$ done

Boundary Nevanlinna-Pick interpolation

More general application: boundary Nevanlinna-Pick interpolation with bounds on angular derivatives

P not uniquely determined by the Stein equation; diagonal entries of P provide bounds on angular derivatives at interpolation nodes on the boundary

Parametrization of solution set in case only $P \succeq 0$

Suppose only $P \succeq 0$. Set $\mathcal{X}^P$ = Hilbert space associated with P (completion of equivalence classes in $\mathcal{X}/\,\mathrm{KerP}$)

Notational sloppiness: $\mathcal{X} = \mathcal{X}^P$
In particular P is well defined on $\mathcal{X}^P$

We assume: $P - T^*PT = E^*E - N^*N$ $(*)$

Then we define an isometry $\mathbf{V}\colon \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$ where
$\mathcal{D}_{\mathbf{V}} = \overline{\mathrm{Ran}} \begin{bmatrix} I_{\mathcal{X}} \\ N \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$, $\mathcal{R}_{\mathbf{V}} = \overline{\mathrm{Ran}} \begin{bmatrix} T \\ E \end{bmatrix} \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ by
$\mathbf{V}\colon \begin{bmatrix} I \\ N \end{bmatrix} x \mapsto \begin{bmatrix} T \\ E \end{bmatrix} x$ for all $x \in \mathcal{X}$

Note that $(*) \Rightarrow \mathbf{V}\colon \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$ is an isometry (with $\mathcal{X}$ equipped with the P metric)

$\mathbf{V}$ is the lurking isometry for this problem!

Alternative characterization of solutions of aAIP

We say that a system matrix $\mathbf{U}\colon \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$ is a minimal unitary-system-matrix extension of $\mathbf{V}$ if

(1) $\mathcal{X}$ is a subspace of $\mathcal{H}$,

(2) $\mathbf{U}|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V} : \mathcal{D}_{\mathbf{V}} \to \mathcal{R}_{\mathbf{V}}$

(3) $\mathcal{X} \subset \mathcal{N} \subset \mathcal{H}$, $\mathcal{N}$ reducing for $\mathbf{U}$ $\Rightarrow \mathcal{N} = \mathcal{H}$

Theorem: characterization of solutions of aAIP
S solves aAIP with admissible data set $\mathcal{D} = (T, E, N, P)$ $\Leftrightarrow$ S has the form
$S(z) = D + zC(I - zA)^{-1}B$ where $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$ is a minimal unitary system-matrix extension of the partially defined isometry $\mathbf{V}$ constructed from $\mathcal{D}$ as above.

In this case the associated map $F^S = \begin{bmatrix} I & -M_S \end{bmatrix} \begin{bmatrix} \mathcal{O}_{E,T} \\ \mathcal{O}_{N,T} \end{bmatrix}$ given by
$F^S(z) = C(I - zA)^{-1}\big|_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{H}(K_S)$

Parametrization of solution set for aAIP

Furthermore, minimal unitary system-matrix extensions of $\mathbf{V}$ given by free-parameter closely connected unitary system matrix $\mathbf{U}_1$ coupled with a universal unitary system matrix $\mathbf{U}_0$ defined as follows:

(1) Universal unitary system matrix determined by $\mathbf{V}$:
Introduce defect spaces $\Delta = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \ominus \mathcal{D}_{\mathbf{V}}$, $\Delta_* = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \ominus \mathcal{R}_{\mathbf{V}}$
Let $\widetilde{\Delta}$ = another copy of Δ, $\widetilde{\Delta}_*$ = another copy of Δ_*
with identificaton maps $\iota \colon \Delta \to \widetilde{\Delta}$, $\iota_* \colon \Delta_* \to \widetilde{\Delta}_*$

Define $\mathbf{U}_0$ by $\mathbf{U}_0 x = \begin{cases} \mathbf{V}x & \text{if } x \in \mathcal{D}_{\mathbf{V}}, \\ \iota(x) & \text{if } x \in \Delta, \\ \iota_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_* \end{cases}$

Identify $\left[\begin{smallmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ and identify $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$

$\Rightarrow$ $\mathbf{U}_0$ decomposes as $\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} \colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}$

Parametrization of the solution set (2)

(2) Free parameter unitary system-matrix: $\mathbf{U}_1$:

$$\mathbf{U}_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \widetilde{\Delta} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \widetilde{\Delta}_* \end{bmatrix}$$

(3) The feedback connection of $\mathbf{U}_0$ and $\mathbf{U}_1$ to get $\mathbf{U}$ = minimal unitary system-matrix extention of $\mathbf{V}_0$:

$\mathbf{U}\colon \begin{bmatrix} x \\ x_1 \\ u \end{bmatrix} \to \begin{bmatrix} \widetilde{x} \\ \widetilde{x}_1 \\ y \end{bmatrix} \Leftrightarrow \exists\ \widetilde{\delta} \in \widetilde{\Delta},\ \widetilde{\delta}_* \in \widetilde{\Delta}_*$ s.t.

$\mathbf{U}_0\colon \begin{bmatrix} x \\ u \\ \widetilde{\delta}_* \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x} \\ y \\ \widetilde{\delta} \end{bmatrix}$ and $\mathbf{U}_1\colon \begin{bmatrix} x_1 \\ \widetilde{\delta} \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{\delta}_* \end{bmatrix}$

Since $U_{33} = 0$ we can solve explicitly:

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} U_{11} + U_{12} D_1 U_{31} & U_{13} C_1 \\ B_1 U_{31} & A_1 \end{bmatrix} & \begin{bmatrix} U_{12} + U13 D_1 U_{32} \\ B_1 U_{32} \end{bmatrix} \\ \begin{bmatrix} U_{21} + U_{23} D_1 U_{31} & U_{23} C_1 \end{bmatrix} & U_{22} + U_{23} D_1 U_{32} \end{bmatrix}$$

Now we want the transfer function $T_{\Sigma_{\mathbf{U}}}(z)$

Parametrization of the solution set (3)

Write $T_{\Sigma_{\mathbf{U}_0}}(z) = \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + z \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - zU_{11})^{-1} \begin{bmatrix} U_{12} & U_{13} \end{bmatrix}$
$=: \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix}$

Write $\mathcal{R}_{\Sigma}[\mathcal{W}] = \Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{W}(z)(I - \Sigma_{22}(z)\mathcal{W}(z))^{-1}\Sigma_{21}(z)$
(Redheffer LFT)

$\Sigma(z) \in \mathcal{S}(\mathcal{Y} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and $\Sigma_{22}(0) = 0 \;\Rightarrow \mathcal{R}_{\Sigma}[\mathcal{W}] \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$
well-defined whenever $\mathcal{W} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$

Calculus of realizations and feedback connections:
$T_{\mathbf{U}}(z) = \mathcal{R}_{\mathbf{U}_0}[T_{\mathbf{U}_1}(z)]$ if $\mathbf{U} = \mathbf{U}_0 \underset{\mathrm{FB}}{*} \mathbf{U}_1$

Set $\mathcal{W} = T_{\mathbf{U}_1} =$ free parameter sweeping $\mathcal{S}(\widetilde{\Delta}, \widetilde{\Delta}_*)$
Conclusion: The set of all solutions of aAIP(T, E, N, P) is given by $S(z) = \mathcal{R}_{\Sigma}(z)[\mathcal{W}(z)]$ where the free parameter $\mathcal{W}(z)$ sweeps $\mathcal{S}(\widetilde{\Delta}, \widetilde{\Delta}_*)$

Part 2: Interpolation problems for functions in vector-valued de Branges-Rovnyak spaces

The $\mathrm{AIP}_{\mathcal{H}(K_S)}$ problem

$\mathrm{AIP}_{\mathcal{H}(K_S)}$-admissible data set $\mathcal{D} = (S, T, E, N, \mathbf{x})$:

- $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, $\mathbf{x} \in \mathcal{X}$
- $T \in \mathcal{L}(\mathcal{X})$, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $N \in Ł(\mathcal{X}, \mathcal{U})$ s.t. $\mathcal{O}_{E,T} \colon \mathcal{X} \to \mathrm{Hol}_{\mathcal{L}(\mathcal{X},\mathcal{Y})}(\mathbb{D})$, $\mathcal{O}_{N,T} \colon \mathcal{X} \to \mathrm{Hol}_{\mathcal{L}(\mathcal{X},\mathcal{U})}(\mathbb{D})$
- $M_{F^S} := \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} \colon \mathcal{X} \to \mathcal{H}(K_S)$ where $F^S(z) = E(I - zT)^{-1} - S(z)N(I - zT)^{-1}$
- $P = M_{F^S}^{[*]} M_{F^S}$ satisfies $P - T^*PT = E^*E - N^*N$ where $[*]$ is adjoint w.r.t. $\mathcal{H}(K_S)$ norm

In case (E, T) is output-stable and $\mathcal{O}_{E,T}^* M_S = \mathcal{O}_{N,T}^*$, then $M_{F^S}^{[*]} M_{F^S} = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} = P$ as before

The $\mathrm{AIP}_{\mathcal{H}(K_S)}$ interpolation problem: Find all $f \in \mathcal{H}(K_S)$ s.t. $M_{F^S}^{[*]} f = \mathbf{x}$ and $\|f\|_{\mathcal{H}(K_S)} \le 1$

Connection with interpolation

If (E, T) is output-stable and we define N by $N^* = \mathcal{O}^*_{E,T} M_S|_{\mathcal{U}}$, then we have seen that

$\mathcal{O}^*_{N,T} = \mathcal{O}^*_{E,T} M_S \colon \mathcal{H}^2_{\mathcal{U}} \to \mathcal{X}$, or $\mathcal{O}_{N,T} = M_S^* \mathcal{O}_{E,T}$
and then $M_{F^S} = \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} = (I - M_S M_S^*) \mathcal{O}_{E,T}$ from which it follows that

$M_F^{[*]} = \mathcal{O}^*_{E,T}|_{\mathcal{H}(K_S)} \Rightarrow M_F^{[*]} f = \mathbf{x}$ amounts to imposing LTOA interpolation conditions on $f \in \mathcal{H}(K_S)$ with a norm constraint: $\|f\|_{\mathcal{H}(K_S)} \leq 1$

Solution criterion for $\text{AIP}_{\mathcal{H}(K_S)}$

$\text{AIP}_{\mathcal{H}(K_S)}$: Find $f \in \mathcal{H}(K_S)$ s.t. $M_{F^S}^{[*]} f = \mathbf{x}$ and $\|f\|_{\mathcal{H}(K_S)} \le 1$
Identify f with $M_f \colon \mathbb{C} \to \mathcal{H}(K_S)$;
Conversely any operator $X \in \mathcal{L}(\mathbb{C}, \mathcal{H}(K_S))$ has the form $X = M_f$ for $f \in \mathcal{H}(K_S)$

$\text{AIP}_{\mathcal{H}(K_S)}$ -problem is: solve the operator equation $M_{F^S}^{[*]} M_f = \mathbf{x}$ for M_f with $\|M_f\| \le 1$

By the Douglas lemma, this is possible $\Leftrightarrow$ $\mathbf{x}\mathbf{x}^* \preceq \mathbf{P} := M_{F^S}^{[*]} M_{F^S}$

Characterization of solutions of AIP$_{\mathcal{H}(K_S)}$-problem

Application of the Douglas-lemma variant gives the following theorem (no use of Stein equation yet):

Theorem: characterization of solutions of AIP$_{\mathcal{H}(K_S)}$-problem

Given an admissible AIP$_{\mathcal{H}(K_S)}$ data set $\mathcal{D} = (T, E, N, S, \mathbf{x})$ together with a prospective solution $f \in \mathcal{H}(K_S)$, we set $\mathbf{P} = M_{F^S}^{[*]} M_{F^S}$. Then TFAE:

(1) f soves the AIP$_{\mathcal{H}(K_S)}$-problem

(2) $\mathbf{K}(z, \zeta) = \begin{bmatrix} 1 & \mathbf{x}^* & f(\zeta)^* \\ \mathbf{x} & P & F^S(\zeta)^* \\ f(z) & F^S(z) & K_S(z,\zeta) \end{bmatrix}$ is a positive kernel on $\mathbb{D}$

(3) $\widehat{\mathbf{P}} := \begin{bmatrix} 1 & \mathbf{x}^* & M_f^{[*]} \\ \mathbf{x} & P & M_{F^S}^{[*]} \\ M_F & M_{F^S} & I_{\mathcal{H}(K_S)} \end{bmatrix} \succeq 0$

Connection with aAIP$_{\mathcal{S}(\mathcal{U},\mathcal{Y})}$-problem

Given a AIP$_{\mathcal{H}(K_S)}$ data set $(T, E, N, S, P, \mathbf{x})$ with $N^* = \mathcal{O}^*_{E,T} M_S|_{\mathcal{U}}$ then (T, E, N, P) is a aAIP-data set and we can consider the aAIP-problem for this data set and there is a Redheffer LFT parametrization for the set of all solutions:

$\mathcal{W} \in \mathcal{S}(\widetilde{\Delta}, \widetilde{\Delta}_*) \mapsto \mathcal{R}_\Sigma(z)[\mathcal{W}(z)]$

Set $G(z) = \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}$,
$\Gamma(z) = U_{21} + G(z)\mathcal{E}(z)U_{31})(I - zU_{11})^{-1}$

Then one can use all this to parametrize solutions of AIP$_{\mathcal{H}(K_S)}$:

- f solves AIP$_{\mathcal{H}(K_S)}$-problem $\Leftrightarrow$ f has the form
  $f(z) = \Gamma(z)\widetilde{\mathbf{x}} + G(z)h(z)$
  where $\mathbf{x} = P^{\frac{1}{2}}\widetilde{\mathbf{x}}$ and $h \in \mathcal{H}(K_S)$ subject to
  $\|h\|_{\mathcal{H}(K_S)} \leq \sqrt{1 - \|\widetilde{\mathbf{x}}\|^2}$

Parametrizatoin continued

- In this case
$\|f\|^2_{\mathcal{H}(K_S)} = \|M_\Gamma \widetilde{\mathbf{x}}\|^2 + \|M_G h\|^2 = \|\widetilde{\mathbf{x}}\|^2 + \|P_{\mathcal{H}(K_\mathcal{E}) \ominus \mathrm{Ker}\, M_G} h\|^2$
and $f_{\min}(z) = \Gamma(z)\widetilde{\mathbf{x}}$
- The problem $\mathrm{AIP}_{\mathcal{H}(K_S)}$ admits a unique solution $\Leftrightarrow$ $\|\widetilde{\mathbf{x}}\| = 1$
or $\overline{\mathrm{Ran}}\, M_F^S = \mathcal{H}(K_S)$

Part 3: Applications

Given inner S, B, M_S is an isometry in $\mathcal{L}(H^2_{\mathcal{U}}, H^2_{\mathcal{Y}})$, $M_S H^2_{\mathcal{U}}$ = the form for a general M_z invariant subspace of $H^2_{\mathcal{Y}}$ (Beurling-Lax)
Set $\mathcal{K}_S = H^2_{\mathcal{Y}} \ominus M_S H^2_{\mathcal{U}}$ (the model space)
Let $B \in \mathcal{S}(\mathcal{W}, \mathcal{Y})$ be another inner funtion
Characterizations of intersections $M_S H^2_{\mathcal{U}} \cap M_B H^2_{\mathcal{W}}$ and $\mathcal{K}_S \cap \mathcal{K}_B$ well known.
Of interest here: $M_{S,B} = \mathcal{K}_S \cap M_B H^2_{\mathcal{W}}$

The space $\mathcal{M}_{S,B} = \mathcal{K}_S \cap M_B H^2_{\mathcal{W}}$

Introduce $T \in \mathcal{L}(\mathcal{K}_B)$, $E \in \mathcal{L}(\mathcal{K}_B, \mathcal{Y})$, $N \in \mathcal{L}(\mathcal{K}_B, \mathcal{U})$ by

- $T\colon h(z) \mapsto \frac{h(z)-h(0)}{z}$ (strongly stable),
- $E\colon h \mapsto h(0)$ ((E, T) output-stable)
- $N\colon h(z) = \sum_{j=0}^{\infty} h_j z^j \mapsto \sum_{j \geq 0} S_j^* h_j$ where $S(z) = \sum_{j \geq 0} S_j z^j$ so $N = \mathcal{O}^*_{E,T} M_S|_{\mathcal{U}}$

$\Rightarrow \mathcal{D} = \{S, E, N, T, \mathbf{x} = 0\}$ is $\text{AIP}_{\mathcal{H}(K_S)}$ is admissible
and $M^{[*]}_{F^S} = \mathcal{O}^*_{E,T}|_{\mathcal{H}(K_S)}$
In this case $\mathcal{O}_{E,T} h)(z) = \sum_{j \geq 0} (ET^j h) z^j = \sum_{j \geq 0} h_j z^j = h(z)$
i.e., $\mathcal{O}_{E,T}$ is the inclusion map $\iota\colon \mathcal{K}_B \to H^2_{\mathcal{Y}}$ and ι^* is the projection $\iota^* = P_{\mathcal{K}(B)}\colon H^2_{\mathcal{Y}} \to \mathcal{K}(B)$

The space $\mathcal{M}_{S,B} = \mathcal{K}_S \cap M_B H^2_{\mathcal{W}}$ continued

Thus for $f \in H^2_{\mathcal{Y}}$ we have $\mathcal{O}^*_{E,T} f = 0 \Leftrightarrow f \in H^2_{\mathcal{Y}} \ominus \mathcal{K}_B = M_B H^2_{\mathcal{W}}$
and $P := M_{F^S}^{[*]} M_{F^S} = \mathcal{O}^*_{e,T}\mathcal{O}_{E,T} - \mathcal{O}^*_{N,T}\mathcal{O}_{N,T}$ amounts to
$P = I_{\mathcal{K}_B} - P_{\mathcal{K}_B} M_S M_S^*|_{\mathcal{K}_B}$

Theorem

Given inner $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and $B \in \mathcal{S}(\mathcal{W}, \mathcal{Y})$, let $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$
come from the associate aAIP$_{\mathcal{S}(\mathcal{U},\mathcal{Y})}$ with admissible data set
(P, T, E, N) as above. Then the space $\mathcal{M}_{S,B}$ is given explicitly as
$\mathcal{M}_{S,B} = G \cdot \mathcal{H}(K_{\mathcal{E}})$ where $\mathcal{E}$ = unique function in $\mathcal{S}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ s.t.
$S = \mathcal{R}_\Sigma[\mathcal{E}]$ and $G(z) = \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}$
Furthermore $M_G \colon \mathcal{H}(K_{\mathcal{E}}) \to \mathcal{M}_{S,B}$ is unitary

Connections with parametrizing kernels of Toeplitz operators, . . .

Thanks!

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