Some results on Q_K spaces

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Origins

- Let $H(\mathbb{D})$ denote the set of analytic functions in the unit disk $\mathbb{D}$ and let $\mathrm{d}m(z)$ be the area measure on $\mathbb{D}$. Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. $f \in H(\mathbb{D})$ is said to be a member of the space $\mathcal{Q}_K$ if

$$||f||_{\mathcal{Q}_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\,\mathrm{d}m(z) < \infty. \qquad (1.1)$$

- Modulo constants, $\mathcal{Q}_K$ is a Banach space under the semi-norm defined in (1.1). For $0 < p < \infty$, $K(t) = t^p$ gives the space $\mathcal{Q}_p$. Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$.
- Note that $\mathcal{Q}_K$ is a Möbius invariant space; that is

$$||f \circ \varphi||_{\mathcal{Q}_K} = ||f||_{\mathcal{Q}_K},$$

where φ is the Möbius transformation of $\mathbb{D}$.

Origins

- From the definition, $\mathcal{Q}_K$ can be viewed as a space generated by a Hilbert spaces $\mathcal{D}_K$, denoted by $\mathcal{Q}_K = \mathcal{M}(\mathcal{D}_K)$; that is

$$\|f\|_{\mathcal{Q}_K} = \sup_{a \in \mathbb{D}} \|f \circ \varphi_a\|_{\mathcal{D}_K},$$

where

$$||f||^2_{\mathcal{D}_K} = \int_{\mathbb{D}} |f'(z)|^2 K\left(\log \frac{1}{|z|}\right) \mathrm{d}m(z) < \infty. \tag{1.2}$$

- For all non-trivial increasing functions K, we have that

$$\mathcal{D} \subset \mathcal{Q}_K \subset \mathcal{B}.$$

- One of the main motivations for the introduction of $\mathcal{Q}_K$ spaces was to understand the gap between BMOA and the Bloch space. For example, there is a function K_0 such that

$$BMOA \subsetneqq \mathcal{Q}_{K_0} \subsetneqq \mathcal{B}.$$

Origins

- A. Aleman and A. Simbotin proved that the space $\mathcal{Q}_K$ equals the Möbius invariant space generated by a (radial) weighted Bergman space $L^2_a(w\mathrm{d}m)$; that is

$$\mathcal{Q}_K = \mathcal{M}(L^2_a(w\mathrm{d}m)), \tag{1.3}$$

where the weight function

$$w(t) = \frac{1}{t^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}K(\log\frac{1}{t}).$$

- $f \in \mathcal{Q}_K$ if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f\circ\varphi_a(z) - f(a)|^2 w(|z|)\,\mathrm{d}m(z) < \infty. \tag{1.4}$$

- Clearly, such function K gives that $BMOA \subsetneq \mathcal{Q}_K \subset \mathcal{B}$.

Origins

- A useful function $\log \frac{1}{1-z} \in \mathcal{Q}_K$ if and only if

$$\int_0^1 K(\log(1/r))(1-r^2)^{-1}\, r\, \mathrm{d}r < \infty. \tag{1.5}$$

- Generally, the space $\mathcal{Q}_K$ is not separable. However, it always contains the following very important separable subspace:

$$\mathcal{Q}_{K,0} = \left\{ f \in H(\mathbb{D}) : \lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, \mathrm{d}m(z) = 0 \right\}.$$

1. M. Essén and H. Wulan, On analytic and meromorphic functions and spaces of $\mathcal{Q}_K$ type, *Uppsala Univ., Dept. Math. Report* 32 (2000), 1-26. *Illinois J. Math.* 46(2002), 1233-1258.
2. M. Essén, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant $\mathcal{Q}_K$ spaces, *J. Funct. Anal.*, 230(2006),78-115.
3. H. Wulan and K. Zhu, Möbius invariant $\mathcal{Q}_K$ spaces, Springer, 2017.

Weight functions

We make the following standing assumptions on the weight function K:

- $K : [0, \infty) \to [0, \infty)$ is non-decreasing and right continuous;
- $K(0) = 0$ and $K(t) = K(1)$ for $t \geq 1$.

Note that $\mathcal{Q}_K$ depends only on the behavior of $K(t)$ near $t = 0$. In many situations we need to consider the following constraints on K:

$$\int_0^1 \frac{\varphi_K(s)}{s}\, \mathrm{d}s < \infty, \tag{A}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^{1+\sigma}}\, \mathrm{d}s < \infty, \qquad 0 < \sigma < 2, \tag{B}$$

where the auxiliary function

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty.$$

Weight functions

- Suppose K satisfies (A) and the doubling condition on $(0, 1)$. Then there exists a weight function K_1 satisfying all standing assumptions such that K_1 is comparable with K on $(0, 1)$ and the function $K_1(t)/t^c$ is increasing on $(0, 1)$ for sufficiently small constant $c > 0$. Conversely, if the function $K(t)/t^c$ is increasing on $(0, 1)$ for some $c > 0$, then K satisfies condition (A).
- Suppose K satisfies (B) for some $\sigma > 0$. Then there exists a weight function K_2 satisfying all the standing assumptions such that K_2 is comparable with K on $(0, 1)$ and the function $K_2(t)/t^{\sigma-c}$ is decreasing on $(0, 1)$ for sufficiently small constant $c > 0$. Conversely, if the function $K(t)/t^{\sigma-c}$ is decreasing for some $0 < c < \sigma$, then K satisfies condition (B) for the same $\sigma > 0$.
- It is clear that the weight function $K(t) = t^p$ satisfies (A) for $0 < p < \infty$ and satisfies condition (B) whenever $0 < p < \sigma$.

K-Carleson measure

- A positive Borel measure μ on $\mathbb{D}$ is said to be a K-Carleson measure provided

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) \mathrm{d}\mu(z) < \infty. \tag{3.1}$$

- A positive Borel measure μ on $\mathbb{D}$ is a K-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K(1-|\varphi_a(z)|^2)\mathrm{d}\mu(z) < \infty. \tag{3.2}$$

K-Carleson measure

- Let K satisfy (A) and (B) for some $\sigma > 0$. Let n be a positive integer. Then $f \in \mathcal{Q}_K$ if and only if
$$|f^{(n)}(z)|^2(1-|z|^2)^{2(n-1)}\mathrm{d}m(z) \tag{3.3}$$
is a K-Carleson measure on $\mathbb{D}$.
- Let K satisfy (A) and (B) for some $\sigma > 0$. Let $\alpha > 1/2$. Then $f \in \mathcal{Q}_K$ if and only if
$$|f^{(\alpha)}(z)|^2(1-|z|^2)^{2(\alpha-1)}\,\mathrm{d}m(z) \tag{3.4}$$
is a K-Carleson measure on $\mathbb{D}$. Here $f^{(\alpha)}$ denotes the α-order derivative of $f \in H(\mathbb{D})$.

K-Carleson measures both on $\mathbb{D}$ and $\mathbb{C}\setminus\overline{\mathbb{D}}$

- Define the Carleson box by

$$S_G(I)=\begin{cases}\{r\zeta\in G:1-\frac{|I|}{2\pi}<r<1,\zeta\in I\}, & G=\mathbb{D},\\ \{r\zeta\in G:1<r<1+|I|,\zeta\in I\}, & G=\mathbb{C}\setminus\overline{\mathbb{D}}.\end{cases}$$

- A positive Borel measure μ on $G=\mathbb{D}$ or $G=\mathbb{C}\setminus\overline{\mathbb{D}}$ is said to be a K-Carleson measure if

$$\sup_{I\subset T}\int_{S_G(I)}K\left(\frac{|1-|z||}{|I|}\right)\mathrm{d}\mu(z)<\infty. \tag{3.5}$$

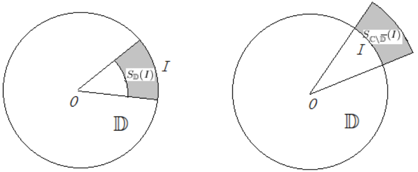

图1 图2

Applications of K-Carleson measures

- For a weight function K, let H^2_K (Morrey type space) denote the space of all functions f in the Hardy space H^2 such that

$$\|f\|_{H^2_K} = \left(\sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f_I|^2 |d\zeta|\right)^{\frac{1}{2}} < \infty, \qquad (4.1)$$

  where

$$f_I = \frac{1}{|I|} \int_I f(\zeta)\, |d\zeta|.$$

- Let K satisfy conditions (A) and (B) for some $\sigma > 0$. If $q \in (0, \sigma)$ is sufficiently close to σ, then $f \in \mathcal{Q}_K$ implies that the fractional derivative $f^{(\frac{1-q}{2})}$ belongs to H^2_K. Conversely, there exists a $q \in (0, \sigma)$ such that $f \in H^2_K$ implies that $f^{(\frac{q-1}{2})} \in \mathcal{Q}_K$.

Applications of K-Carleson measures

- If K satisfies (A) and (B) for some $\sigma > 0$, then $f \in \mathcal{Q}_K$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f\circ\varphi_a(z) - f\circ\varphi_a(w)|^2}{|1-z\overline{w}|^4} K(1-|z|^2)\mathrm{d}m(z)\mathrm{d}m(w) < \infty. \tag{4.2}$$

- If K satisfies (A) and (B) for some $\sigma > 0$, then $f \in \mathcal{Q}_K$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f\circ\varphi_a(z) - f\circ\varphi_a(w)|^2}{|1-z\overline{w}|^4} K^{1/2}(1-|z|^2)K^{1/2}(1-|w|^2)\mathrm{d}m(z)\mathrm{d}m \tag{4.3}$$

- If K satisfies (A) and (B) for some $\sigma > 0$, then $f \in \mathcal{Q}_K$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\partial\mathbb{D}} |f(\zeta)|^2 \,\mathrm{d}\mu_z(\zeta) - |f(z)|^2 \right) \frac{K(1-|\varphi_a(z)|^2)}{(1-|z|^2)^2} \,\mathrm{d}m(z) < \infty, \tag{4.4}$$

where

$$\mathrm{d}\mu_z(\zeta) = \frac{1-|z|^2}{2\pi|\zeta - z|^2}|\mathrm{d}\zeta|, \quad z\in\mathbb{D}, \quad \zeta\in\partial\mathbb{D}.$$

Problem 1: BMOA

- For all nontrivial K, we have

$$\mathcal{D} \subset \mathcal{Q}_K \subset \mathcal{B}.$$

- $\mathcal{Q}_K = \mathcal{D}$ if and only if $K(0) > 0$.
- $\mathcal{Q}_K = \mathcal{B}$ if and only if

$$\int_0^1 K(\log(1/r))(1-r^2)^{-2}\, r\, \mathrm{d}r < \infty.$$

- $\mathcal{Q}_K = BMOA$ if and only if $K = ?$

Problem 1: BMOA

- If K satisfies (B) for $\sigma = 1$, then $\mathcal{Q}_K \subset BMOA$. Convesely, it is not true since there exists a function K satisfying (B) for $\sigma > 1$ such that $\mathcal{Q}_K \subset BMOA$. For example, choose $K(t) = t(\log \frac{e}{t})^p$ for $p > 0$.
- K is convex $\Longrightarrow BMOA \subset \mathcal{Q}_K$.
- K is concave $\Longrightarrow \mathcal{Q}_K \subset BMOA$.

Theorem (G. Bao, J, Mashreghi, S. Pouliasis and W, 2017)

If K is concave, then $\mathcal{Q}_K = BMOA$ if and only if

$$\int_0^1 [K'(t) - (1-t)K''(t)]dt < \infty.$$

Problem 1

Find a sufficient and necessary condition on K such that

$$\mathcal{Q}_K = BMOA.$$

Problem 2: Multiplier Spaces

- Let $M(X)$ be the multiplier of a Banach space X; that is,

$$M(X) = \{g \in H(\mathbb{D}) : fg \in X \text{ for all } f \in X\}.$$

- $M(H^2) = H^\infty$.
- $M(\mathcal{B}) = H^\infty \bigcap \mathcal{B}^{\log}$.
- $M(\mathcal{Q}_p) = H^\infty \bigcap \mathcal{Q}_p^{\log}$, $0 < p \leq 1$.
- $M(\mathcal{D}_p) = H^\infty \bigcap \mathcal{W}_p$, $0 < p < 1$.
- $M(\mathcal{Q}_K) =?$

Problem 2: Multiplier Spaces

- We conjecture that

$$M(\mathcal{Q}_K) = H^\infty \bigcap \mathcal{Q}_K^{\log},$$

where

$$\|f\|^2_{Q_K^{\log}} = \sup_{a\in\mathbb{D}} \left(\log\frac{2}{1-|a|^2}\right)^2 \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)dA(z) < \infty.$$

- If we consider Blaschke products

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|}\frac{z-z_n}{1-\overline{z_n}z},$$

where $\{z_n\}$ is the sequence of zeros of B with satisfying the condition $\sum_{n=1}^{\infty}(1-|z_n|^2) < \infty$, our conjecture is true; that is

Problem 2: Multiplier Spaces

Theorem (Li and W, 2021)

Let K satisfy (A) and (B) for $\sigma = 1$ and let B be a Blaschke product with zeros $\{z_n\}$. The following statements are equivalent:

(i) $B \in M(\mathcal{Q}_K)$;

(ii) $B \in H^\infty \bigcap \mathcal{Q}_K^{\log}$;

(iii)

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|^2} \right)^2 \sum_{n=1}^{\infty} K(1 - |\varphi_a(z_n)|^2) < \infty.$$

Problem 2

$$M(\mathcal{Q}_K) = ?$$

Problem 3: The Corona Theorem

- L. Carleson (1962) for $M(H^2) = H^\infty$.
- V. Tolokonnikov (1991) for $M(\mathcal{D})$ and $M(BMOA)$.
- A. Nicolau and J. Xiao (1997) for $H^\infty \bigcap \mathcal{Q}_p, 0 < p < 1$.
- J. Pau (2008) for $M(\mathcal{Q}_p), 0 < p < 1$.
- J. Pau (2008) for $H^\infty \bigcap \mathcal{Q}_K$.
- Li and W (2021) for $M(\mathcal{Q}_K)$.

Problem 3: The Corona Theorem

Theorem (Li and W, 2021)

Let K satisfy (A) and (B) for $\sigma = 1$. If $f_1, f_2, \ldots, f_n$ in $M(\mathcal{Q}_K)$ with

$$0 < \delta \leq |f_1(z)| + \cdots + |f_n(z)|, \ z \in \mathbb{D},$$

then there exist $g_1, g_2, \ldots, g_n$ in $M(\mathcal{Q}_K)$ such that

$$f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) = 1.$$

Problem 3

Let K satisfy (A) and (B) for some $\sigma > 1$. Does the Corona Theorem for $M(\mathcal{Q}_K)$ still hold?

Pronlem 4: Boundary $\mathcal{Q}_K$ spaces

- $f \in L^2$ is said to be a $\mathcal{Q}_K(\partial\mathbb{D})$ function if

$$||f||^2_{\mathcal{Q}_K(\partial\mathbb{D})} := \sup_{I\subset\partial\mathbb{D}} \int_I \int_I \frac{|f(\zeta)-f(\eta)|^2}{|\zeta-\eta|^2} K\left(\frac{|\zeta-\eta|}{|I|}\right) |\mathrm{d}\zeta||d\eta| < \infty. \tag{4.5}$$

- If K satisfies (A) and (B) for some $\sigma = 1$, then

$$\mathcal{Q}_K = \mathcal{Q}_K(\partial\mathbb{D}).$$

Problem 4

Let K satisfy (A) and (B) for some $\sigma > 1$.

$$\mathcal{Q}_K = \mathcal{Q}_K(\partial\mathbb{D})?$$

Thank you for your attention!