Dynamical sampling, frames, and inverse problems

Akram Aldroubi

Vanderbilt University

Function Spaces and their Applications: Riesz Bases, frames, and Signal Processing

Supported by NSF/DMS

Motivation: Martin Vetterli's and Yue Lue's work John Murray-Bruce, Pier Luigi Dragotti, Martin Vetterli, Yue Lu, ...on wireless sensing network.

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Collaborators: Roza Aceska, Victor Bailey, Carlos Cabrelli, Jacqueline Davis, Karlheinz Gröchenig, Longxiu Huang, Keri Kornelson, Ilya Krishtal, Akos Ledeczi, Phillip Jaming, Roy Lederman, Ursula Molter, Armenak Petrosyan, José-luis Romero, Sui Tang, Peter Volgyesi, and Eric Weber.

Dynamical sampling

Let $H : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Recover $H(t, x)$ and/or to find various parameters that govern the relation between u and H from a set of samples $\{u(t_i, x_j)\}$ of a function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ on a spatio-temporal set $\{(t_i, x_j)\}$.

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Consider the abstract Initial Value Problem (IVP) in $\mathcal{H}$:

$$\begin{cases} \dot{u}(t) = Au(t) + F(t) \\ u(0) = u_0, \end{cases} \qquad t \in \mathbb{R}_+, \;\; u_0 \in \mathcal{H}, \tag{1}$$

where $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ generates a strongly continuous semigroup T ($T(t) = e^{tA}$ when A is bounded).

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Unknown to be recovered: u_0, A, F, etc.

Dynamical sampling

$$\begin{cases} \dot{u}(t) = Au(t) + F(t) \\ u(0) = u_0, \end{cases} \qquad t \in \mathbb{R}_+,\ u_0 \in \mathcal{H}. \tag{2}$$

Various Problems: Let $u(t_i, x_j)$ be measurements samples on a sampling pattern $\{(t_i, x_j)\}$.

- Space-time trade-off type problems: e.g., A known, $F = 0$, find u_0.

- System identification type problems: e.g., $F = 0$, u_0 and A unknowns. Find A or some of its characteristics.

- Source term type problems: e.g., A is known, find F.

Definitions

Let $\mathcal{G} \subset \mathcal{H}$ countable, $\mathcal{T} \subset \mathbb{R}$, and $E = \{A^t g : g \in \mathcal{G}, t \in \mathcal{T}\} \subset \mathcal{H}$.

Definitions

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E is frame for $\mathcal{H}$ if there exists $C_1, C_2 > 0$ such that

$$C_1 \|f\|^2 \leq \sum_{g \in \mathcal{G}} \int_{\mathcal{T}} |\langle f, A^t g\rangle|^2 d\mu(t) \leq C_2 \|f\|^2.$$

If $\tau = [0, L]$ and μ is the Lebesgue measure we say that E is a semi-continuous frame.

If μ is discrete, $C_1 \|f\|^2 \leq \sum_{g \in \mathcal{G}} \sum_{t_j \in \mathcal{T}} |\langle f, A^{t_j} g\rangle|^2 \leq C_2 \|f\|^2$.

E is a Riesz basis if $A\|c\|^2 \leq \| \sum_{j,g} c_{j,g} A^{t_j} g\|^2 \leq B\|c\|^2$, for all $c \in \ell^2$

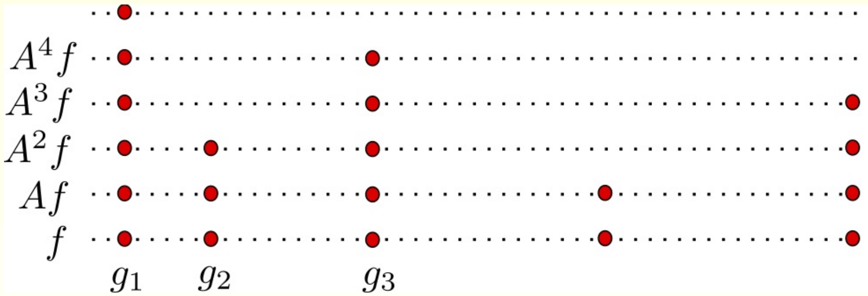
Frames or Riesz bases provide unconditional rerpresentations of functions (stable reconstruction).

Space-time tradeoff

Example: If $f \in \mathcal{H} = \ell^2(\mathbb{Z})$ is an initial distribution that is evolving in time under the action of an evolution operator A,

$$f_n = A^n f, \tag{3}$$

Find conditions on A, $I \subset \mathbb{Z}$ and l_i, such that any $f \in \mathcal{H}$ can be recovered, in a stable way, from the samples $\mathcal{D} = \{f(i), (Af)(i), \ldots, (A^{l_i})f(i) : i \in I\}$.



Problem: Space-time trade-off

Let $A \in B(\mathcal{H})$, $\mathcal{G} \subset \mathcal{H}$, and $Y = \{\langle A^t f, g\rangle : g \in \mathcal{G}, t \in \mathcal{T}\}$ a set of measurements of $f \in \mathcal{H}$.

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Every $f \in \mathcal{H}$ can be recovered in a stable way from Y if and only if $\{(A^*)^t g :\ g \in \mathcal{G}, t \in \mathcal{T}\}$ is a frame for $\mathcal{H}$.

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Problem: Find conditions on $A, \mathcal{G}, \mathcal{T}$ such that $\{A^t g :\ g \in \mathcal{G}, t \in \mathcal{T}\}$ is a frame for $\mathcal{H}$.

Case: $\{A^n g : \; g \in \mathcal{G}, n \geq 0\}$

Theorem. *[A., Cabrelli, Cakmak, Molter, Pertrosyan- JFA 2017] If A is a normal operator on $\mathcal{H}$ then, for any set of vectors $\mathcal{G} \subset \mathcal{H}$, the system of iterates $\{A^n g\}_{g \in \mathcal{G}, n \geq 0}$ is not a basis for $\mathcal{H}$.*

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Conjecture does not hold if operator is not normal. e.g., for shift operatornon $\ell^2(\mathbb{N})$ $(S(x_1, x_2, \dots) = (0, x_1, x_2, \dots))$, $\{S^n e_1\}$ is an orthonormal basis for $\ell^2(\mathbb{N})$, where $e_1 = (1, 0, \dots)$.

Frames induced by a single vector: $|\mathcal{G}| = 1$

Theorem. *[A., Cabrelli, Molter, Tang- ACHA 2017] (with help from J. Antezana)*
Let $A \in B(\mathcal{H})$ *be normal,* $\dim(\mathcal{H}) = \infty$. *Then,* $\{A^n g\}_{n \geq 0}$ *is a frame for* $\mathcal{H}$ *if and only if*

1. $A = \sum\limits_{j \in \mathbb{N}} \lambda_j P_j$, *where* P_j *are rank one ortho-projections.*
2. $|\lambda_k| < 1$ *for all* k, *and* $|\lambda_k| \to 1$ *and* $\{\lambda_k\}$ *satisfies Carleson condition* $\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta$ *for some* $\delta > 0$.
3. $0 < C_1 \leq \frac{\|P_j g\|}{\sqrt{1 - |\lambda_j|^2}} \leq C_2 < \infty$ *for some constants* C_1, C_2.

Frames induced by a finitely many vectors: $|\mathcal{G}| < \infty$

Theorem. *[A., Petrosyan– 2017] If for a normal operator $A \in B(\mathcal{H})$ in an infinite dimensional space $\mathcal{H}$ the system of vectors $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ with $|\mathcal{G}| < \infty$ is a frame, then A is unitarily equivalent to an operator $\Lambda = \sum_j \lambda_j P_j$ where P_j are projections such that $\dim P_j \leq |\mathcal{G}|$, $|\lambda_k| < 1$ for all k, and $|\lambda_k| \to 1$.*

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Construction of a frame similar to previous theorem when $|\mathcal{G}| < \infty$: see Cabrelli, Molter, Paternostro, Philipp $|\mathcal{G}| < \infty$, J. Anal. Math. 2020.

Source term recovery: problem statement

We consider an abstract IVP of the form

$$\begin{cases} \frac{\partial}{\partial t}u(t) = Au(t) + f(t) + \eta(t) \\ u(0) = u_0. \end{cases} \tag{4}$$

$\eta : \mathbb{R}_+ \to \mathcal{H}$ is Lipschitz continuous and models a background source, and $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a generator of a C_0-semigroup $T(t)$.

The burst-like source term f is of the form

$$f(t) = \sum_{j=1}^{N} f_j \delta(t - t_j),$$

for some unknowns $N \in \mathbb{N}$, $0 = t_0 < t_1 \ldots < t_N$, and $f_j \in V \subset \mathcal{H}$.

Problem statement – continued

$$\begin{cases} \frac{\partial}{\partial t} u(t) = Au(t) + f(t) + \eta(t) \\ u(0) = u_0. \end{cases} \tag{5}$$

Problem. *Construct a countable set* $\mathcal{G} \subset \mathcal{H}$*, and time step* $\beta \in \mathbb{R}^+$ *that allow one to stably and accurately approximate* $f(t) = \sum_{j=1}^{N} f_j \delta(t - t_j)$*, from the noisy samples (additive noise* ν*)*

$$\mathfrak{m}_n(g) = \langle u(n\beta), g \rangle + \nu(n\beta, g), \quad n \in \mathbb{N},\ g \in \mathcal{G}$$

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where the background source η is Lipschitz with Lipschitz constant L, and the noise ν is bounded $\sup\limits_{n,g} |\nu(n\beta, g)| \le \sigma$.

Predictive Algorithms: Discrete sampling

Idea of algorithms: Pick a set $\widetilde{\mathcal{G}}$ which a frame for V (recall $f(t) = \sum_{j=1}^{N} f_j \delta(t - t_j)$, where $f_j \subset V \subset \mathcal{H}$). Now choose a set of sampling functional $\mathcal{G} = \widetilde{\mathcal{G}} \cup T^*(\beta)\widetilde{\mathcal{G}}$, where β is a time step, ($T(t)$ is the semi-group generated by A).

Key observation: At time $t_n = n\beta$, we get two measurements $\mathfrak{m}_n(\tilde{g}) = \langle u(t_n), \tilde{g}\rangle$, and $\mathfrak{m}_n(T^*(\beta)\tilde{g}) = \langle u(t_n), T^*(\beta)\tilde{g}\rangle$.

$$\mathfrak{m}_n(\tilde{g}) = \sum_{t_j < n\beta} \left\langle T(n\beta - t_j) f_j, \tilde{g} \right\rangle + \int_0^{n\beta} \left\langle T(n\beta - \tau)\eta(\tau), \tilde{g} \right\rangle d\tau + \nu(n\beta, \tilde{g}).$$

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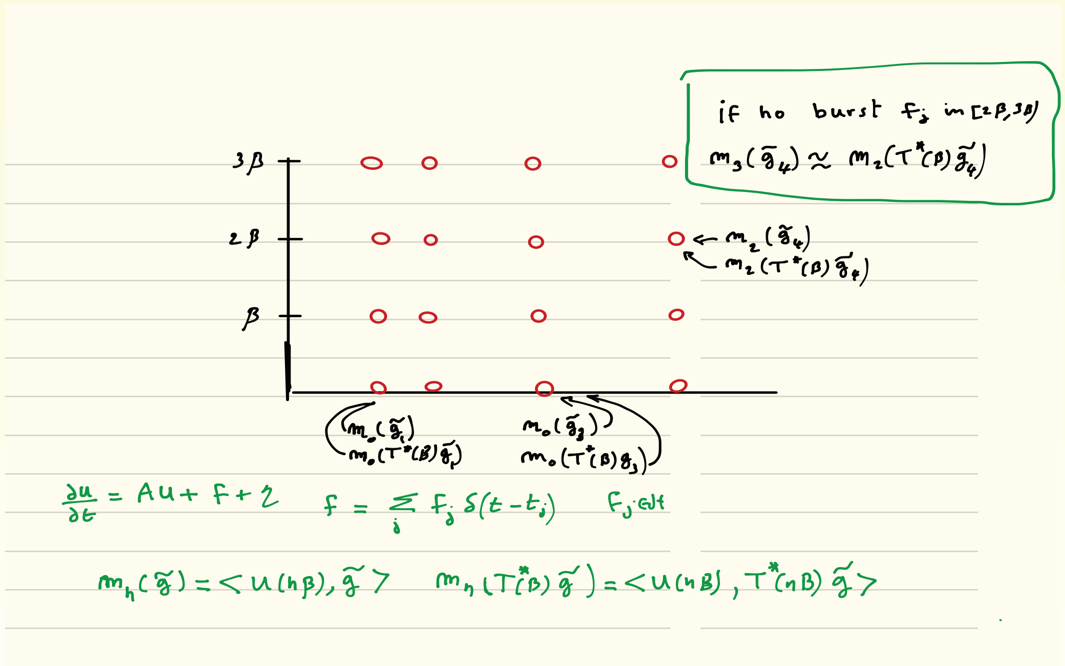
The measurement $\mathfrak{m}_n(T^*(\beta)\tilde{g})$ predicts (up to η, ν) $\mathfrak{m}_{n+1}(\tilde{g}) = \langle u(t_{n+1}), \tilde{g} \rangle$ at t_{n+1} if no burst f_j occurs with $t_j \in [t_n, t_{n+1})$.

i.e., $\mathfrak{m}_n(T^*(\beta)\tilde{g}) - \mathfrak{m}_{n+1}(\tilde{g}) \approx 0$ if no burst in $[n\beta, (n+1)\beta)$.



Predictive Algorithms: Discrete sampling

Let $f(t) = \sum_{j=1}^{N} f_j \delta(t - t_j)$, and assume that $t_{j+1} - t_j > \gamma$ (γ is arbitrary). Choose β with $3\beta < \gamma$, and let $\Gamma_n = \mathfrak{m}_{n+1}(g) - \mathfrak{m}_n(T^*(\beta)g)$. If there is no burst in $[n\beta, (n+2)\beta)$ we get

$$\begin{aligned} |\Gamma_{n+1} - \Gamma_n| &\leq \int_0^\beta \|\eta((n+1)\beta + \tau) - \eta(n\beta + \tau))\| \|T^*(\beta - \tau)g\| d\tau + 4\sigma \\ &\leq CL\beta^2 \|g\| + 4\sigma. \end{aligned} \tag{6}$$

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Thus, we declare that a burst occurred in $[n\beta, (n+2)\beta)$ if

$$|\Gamma_{n+1} - \Gamma_n| \ge K \left(CL\beta^2 \|g\| + 4\sigma \right) = Q$$

where K is some chosen number larger than 1.

Predictive Algorithms: Discrete sampling

Burst detector function $\mathfrak{f}(g) \approx \langle f_j, g\rangle$ if $t_j \in [(n+1)\beta, (n+2)\beta)$ as follows

$$\mathfrak{f}(g) = \begin{cases} \Gamma_{n+1} - \Gamma_n, & \text{if } |\Gamma_{n+1} - \Gamma_n| \geq Q \quad \text{and} \quad |\Gamma_{n+2} - \Gamma_{n+1}| \geq Q; \\ \\ 0, & \text{otherwise.} \end{cases} .$$

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Theorem. *[A., Kornelson, Krishtal, Huang] Assume that* $t_{j+1} - t_j > \gamma$ *in the burst term* $f(t) = \sum_{j=1}^N f_j \delta(t - t_j)$*, and* $\|(T^*(\beta) - I)g\| \le D(\beta)\|g\|$*, for* $g \in \mathcal{G}$*, where* $D(\beta) \to 0$ *as* $\beta \to 0$*. Then for any sufficiently small* $\beta > 0$ *the burst term* f *is well approximated by* $\tilde{f}(t) = \sum_{j=1}^N \tilde{f}_j \delta(t - \tilde{t}_j)$ *that is obtained via a predictive algorithm. In particular* $|t_j - \tilde{t}_j| \le \frac{\beta}{2}$ *as long as* $\tilde{f}_j \neq 0$*, and*

$$\|\tilde{f}_j - f_j\| \le C_1 L \beta^2 + C_2 \sigma + C_3 D(\beta) \|f_j\|$$

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$$\|\tilde{f}_j - f_j\| \leq C_1 L \beta^2 + C_2 \sigma + C_3 D(\beta)\|f_j\|$$

Another predictive Prony-like algorithm allows us to estimate $|t_j - \tilde{t}_j|$ to within $O(\beta^2)$.

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Thank you