Exponential Frames and Riesz Bases

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Based on joint works with G.Kozma, S.Nitzan, and A.Ulanovskii

1 Introduction

Let H be a separable Hilbert space. We are interested in systems of vectors $\{u_\lambda\}$ in H which have good approximation or representation properties.

In this talk we consider

$$H = L^2(S),\ \ S \subset \mathbb{R},\ \ 0 < |S| < \infty,\ \ \{u_\lambda\} = E(\Lambda),$$

where $E(\Lambda)$ is the set of exponentials with frequencies in Λ:

$$E(\Lambda) := \{e^{2\pi i\lambda t},\ \ \lambda \in \Lambda \subset \mathbb{R}\}.$$

Certainly, the orthogonal bases are the best. However, such a basis often can not be found in a required form. In particular, the classical Fourier system $E(\mathbb{Z})$ is an ONB in L^2 on $[0,1]$. But an orthogonal basis may not exist even for a union of two intervals, for example for $S = [0,2] \cup [3,5]$.

So, one needs a generalization of an orthogonal system, which keeps its good properties but is more available for constructions.

2 Riesz bases

Definition. A system of vectors $\{u_\lambda\}$ is called a *Riesz basis* (RB) for H if it is the image of an ONB under a linear isomorphism of the space.

Equivalently, a system $\{u_\lambda\}$ is an RB if

(1) it is complete in H;

(2) there are positive constants A, B such that for any (finite) sequence of numbers $\{c(\lambda)\}$ the inequality holds:

$$A\|c(\lambda)\|^2 \le \|\sum_\lambda c(\lambda)u_\lambda\|^2 \le B\|c(\lambda)\|^2. \tag{1}$$

One can see that such a system is a basis in Schauder's sense, i.e. every vector $f \in H$ can be represented by a series $f = \sum_\lambda c(\lambda)u_\lambda$, and the representation is unique.

A natural question: is every (normed) Schauder basis in H a Riesz basis? The negative answer is due to K. Babenko (1949). His example is based on the system $E(\mathbb{Z})$ in a weighted L^2 space. We notice that Babenko's weight belongs to an important class of weights discovered by B. Muckenhoupt (1972).

It should be noticed also that this class of weights was independently discovered by Moscow student Alexander Krantsberg (1972), who characterized weighted L^p spaces in which the classical Haar system is a Schauder basis.

Two important results on exponential RBs:

(i) The Landau density theorem. How many exponentials one needs to get an RB for $L^2(S)$? H. Landau proved the following necessary condition: *if $E(\Lambda)$ is an RB for $L^2(S)$, then $D(\Lambda) = |S|$.*

I recall that a discrete set Λ has Beurling uniform density D if

$$\#\{\lambda \in [a, a + r]\} = Dr + o(r)\,,$$

where "o" is uniform w.r.t. a.

(ii) The Paley-Wiener stability theorem: *If $E(\Lambda)$ is an RB for $L^2(S)$, then there is a $\delta = \delta(S, L) > 0$ such that for any set Λ', $\|\Lambda - \Lambda'\|_\infty < \delta$, the system $E(\Lambda')$ is also an RB.*

In the classical case $S = [0, 1]$, $\Lambda = \mathbb{Z}$, the number δ can be taken 1/9 (Paley-Wiener). The sharp value is 1/4 (Kadec).

RB are more available for construction than ONB. K. Seip (1993) proved that if S is a union of two intervals, then an exponential RB does exist.

Key lemma: *for every interval I, $|I| < 1$, there is an RB $E(\Lambda)$, $\Lambda \subset \mathbb{Z}$.*

G. Kozma and S. Nitzan (2015) proved that an exponential RB exists for any finite union of intervals.

However, it remained unknown during a long time whether an exponential Riesz basis in $L^2(S)$ exists for an *arbitrary* bounded set S.

Here is the answer:

Theorem 1 (G. Kozma, S. Nitzan, A. O., 2021) *There exists an open (closed) bounded set S with no Riesz basis $E(\Lambda)$ for $L^2(S)$.*

3 Frames

The concept of frame, introduced by Duffin and Schaeffer (1952) is important in our context.

Definition. A system of vectors $\{u_\lambda\}$ is called *a frame* in H if there are positive constants A, B (frame bounds) such that for any $f \in H$ the inequality holds:

$$A\|f\|^2 \leq \sum_\lambda |\langle f, u_\lambda\rangle|^2 \leq B\|f\|^2 . \tag{2}$$

The right inequalities in (1) and (2) are equivalent. They both are called Bessel's inequality. However, the two others are dual.

For an exponential system $E(\Lambda)$ in $L^2(S)$, the left inequality in (2) provides the stable reconstruction of a function $F \in L^2(\mathbb{R})$ with the spectrum in S from its sampling on Λ.

On the other hand, the left inequality in (1) is responsible for the interpolation of a discrete l^2-function on Λ by an L^2-function F with spectrum in S.

H. Landau proved a necessary condition for exponential frames: *if* $E(\Lambda)$ *is a frame on* S, *then*

$$\liminf_{r\to\infty} \#\{\lambda \in [a, a+r]\}/r \geq |S|,$$

uniformly on a.

Frame decomposition. The most important property of frames is that every vector f in H admits a decomposition into a series

$$f = \sum_\lambda c(\lambda)\, u_\lambda$$

with l^2 - coefficients. In general, the decomposition is not unique. In particular, one has a Fourier type formula

$$f = \sum \langle G^{-1} f, u_\lambda\rangle\, u_\lambda,$$

where G is a bijective operator,

$$f \to Gf = \sum \langle f, u_\lambda\rangle\, u_\lambda.$$

Construction. For a bounded set S it is easy to construct an exponential frame. It suffices to choose $\Lambda = (1/\text{diam}(S))\mathbb{Z}$.

On the other hand, the following problem remained open during a long time: *does a frame $E(\Lambda)$ exist for every unbounded set S of finite measure?*

Observe that if S is unbounded, then functions in the Paley-Wiener space

$$PW_S := \{F = \hat{f},\ f \in L^2(S)\}$$

may loose the analyticity property, even the smoothness property, keeping only the continuity. So it seems to be much more difficult to recover F from a discrete sampling.

A sampling set Λ providing the uniqueness of recovering was constructed in

Theorem 2 (A. O., A. Ulanovskii, 2011) *Let $S \subset \mathbb{R}$ be an (unbounded) set of finite measure. Then there is a set of frequencies Λ, $D(\Lambda) = |S|$, such that the system $E(\Lambda)$ is complete in $L^2(S)$.*

A few years later we proved a stronger, stable sampling theorem:

Theorem 3 (S. Nitzan, A. O., A. Ulanovskii, 2016) *For every (unbounded) set $S \subset \mathbb{R}$ of finite measure, the space $L^2(S)$ admits an exponential frame.*

While Theorem 2 was obtained by a direct construction, the proof of Theorem 3 is based on a deep result by Marcus, Spielman, and Srivastava.

4 "Good" frames

The ideas which led us to the proof of Theorem 3 are connected to the following problem:

Let A be an orthogonal matrix of order N. Choose any k columns, $k \ll N$. Can one find, say, $2k$ rows of the matrix A so that the obtained $2k \times k$ submatrix would be well-invertible?

Problems of this kind have been considered by B. Kashin, J. Bourgain - L. Tzafriri, and other authors. A great progress in the area achieved in the outstanding paper by A. Marcus, D. Spielman, and N. Srivastava (2015).

Based on their results we proved the following

Lemma 1 *Let M be a $N \times k$ matrix composed by k columns of some orthogonal matrix of order N. Assume that all rows of M have the same norm. Then there is a subset $J \subset \{1, 2, ..., N\}$ such that*

$$c\frac{k}{N}\|w\|^2 \leq \|M(J)\,w\|^2 \leq C\frac{k}{N}\|w\|^2, \quad w \in \mathbb{C}^k.$$

Here $M(J)$ is the corresponding sub-matrix of M, and $C > c > 0$ are some absolute constants.

We use the lemma for construction of "good" frames. We say that a frame $E(\Lambda)$ is "good" if the magnitude of the frame bounds is comparable with $|S|$.

This property is similar to what we have for ONBs on S. It characterizes the quality of the frame decomposition.

Let us illustrate the concept of good frame by a simple example. Take $S = [0, 1/N]$ with a large N. Then $E(\mathbb{Z})$ is a frame. But its density much larger then $|S|$, so the frame is considerably overcomplete.

On the other hand, by taking $\Lambda = N\mathbb{Z}$ we get an ONB on S.

Can one get such a good frame for any set S?

Using Lemma 1 we gave a positive answer. For any bounded set S we constructed an exponential frame with the frame bounds satisfying the condition

$$A \geq c|S|, \ B \leq c'|S|\,,$$

where $c' > c > 0$ are absolute constants.

Also, we extended this results to unbounded sets of finite measure, which gives Theorem 3.

5 Comments on the proof of Theorem 1

Consider a version of the problem for a weighted L^2-space on $\mathbb{R}$.

Claim. There is no exponential RB for the space.

Indeed, if $\{e_\lambda\}$ is such a basis, we decompose the function $f(t) := 1_{[0,1]}$ into a series $\sum c_\lambda e_\lambda(t)$. Then consider the translate

$$f_h(t) = f(t-h).$$

Formal calculation (which is easy to justify) gives

$$f_h(t) = \sum_\lambda c_\lambda e^{-2\pi i\lambda h}\, e_\lambda(t). \tag{3}$$

Observe that $\|f_h\| \to 0$ $(h \to \infty)$, but the l_2-norm of the coefficients is fixed. This gives a contradiction.

I used this simple argument many years ago. It does not work directly for $L^2(S)$.

However, there is a nontrivial way to perform it.

6 Open problems

– To characterize sets S which admit an exponential RB. This problem is open even for ONB. (Recall the Fuglede conjecture).

– Theorem 1 shows that exponential Riesz bases may not work for representation of L^2-functions on bounded sets. Notice that RB = "minimal frame". In this context one may ask: given S, can one find a frame $E(\Lambda)$ such that $D(\Lambda) = |S|$?

– Does every space $L^2(S)$ admit a complete and minimal system $E(\Lambda)$?

– At last, the following problem is well-known: does a disc in $\mathbb{R}^2$ admit an exponential RB?