ISP: brief overview of recent developments

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I Introduction

$\mathcal{X}$ Banach space (complex, separable infinite dim'l)

$\mathcal{L}(\mathcal{X}) =$

For $T \in \mathcal{L}(\mathcal{X})$ $\operatorname{Lat} T = \{\mathcal{M}$ c.l.s $| \ T\mathcal{M} \subset \mathcal{M}\}$

$\boxed{\text{ISP: Given any } \mathcal{X} \text{ and any } T \in \mathcal{L}(\mathcal{X}) \text{ does } \operatorname{Lat} T \neq \{(0), \mathcal{X}\}\text{?}}$

A bit of history:

.. Problem formalized as above more or less around the middle of the previous century.

.. Promising results towards a positive answer '54 AS '66 R '73 L

but in '75 ENFLO announced a negative

* answer i.e. $\exists \mathcal{X}, \exists T \in \mathcal{L}(\mathcal{X})$ $\operatorname{Lat} T = \{(0), \mathcal{X}\}$.

Checking this result took several years and is mostly due to Beauzamy

.. in '84 READ produced a simpler example and a year later a still simpler one:

Read '85 $\exists T \in \mathcal{L}(\ell^1)$, $\operatorname{Lat} T = \{(0), \ell^1\}$

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ISP($\mathcal{C}$) remains open for most "classes" of op.

Comments and additional terminology

1. Of course the ISP is not only a goal in itself; it is strongly motivated by the search of better knowledge of operators.

Illustration with the concept of reflexivity (of operators)

|| Dfn $\mathrm{Alg}(\mathrm{Lat}\, T) = \{B \in \mathcal{L}(\mathcal{H});\ \mathrm{Lat}\, B \supset \mathrm{Lat}\, T\}$
|| T is reflexive if $\mathrm{Alg}(\mathrm{Lat}\, T) = \mathcal{W}_T$

Some results and comments

2. cyclicity For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$ let

$$\mathcal{O}_x = \{x, Tx, \dots, T^n x, \dots\}$$

Dfn x is cyclic for T if

$$[\mathcal{O}_x] \text{ (denoted also } \bigvee_{n \geq 0} T^n x) = \mathcal{H}$$

Thus

$$\mathrm{ISP}(T) \iff \exists\, x \in \mathcal{H} \setminus \{0\},\ x \text{ non cyclic for } T$$

3 Stronger notion

x is hypercyclic for T if

$$\overline{\mathcal{O}_x} = \mathcal{H}$$

leads to "I Subset P":

Given any $T \in \mathcal{L}(\mathcal{H})$ does there exist a closed subset E in $\mathcal{H}$ such that

$(E \notin \{\phi, \{0\}, \mathcal{H}\})$ $TE \subset E$

READ: NO for certain $\mathcal{H}$

II ~~Positive~~ Some answers to $ISP(\mathcal{X})$ for certain $\mathcal{X}$.

* Agyros Haydon (2011)

$\exists\ \mathcal{X}\quad \mathcal{L}(\mathcal{X}) = \mathbb{C}I + \mathcal{K}(\mathcal{X})$

Corollary On such $\mathcal{X}$ any T has i.s.

Comment.

- strictly singular op. (SSO) are natural generalizations of compact op.

Results of Gowers - Maurey ('93 and '97) show:

$\exists\ \mathcal{X}\quad \mathcal{L}(\mathcal{X}) = \{\mathbb{C}I + SSO(\mathcal{X})\}$

but

Read ('99): $\exists\ \mathcal{X}\ \exists\ T \in SSO(\mathcal{X})\ \mathrm{Lat}\,T = \{(0), \mathcal{X}\}$

Other results

* 1983 <u>Atzmon</u> There exists a nuclear Frechet space $\mathcal{F}$ and $T \in \mathcal{L}(\mathcal{F})$ such that $\mathrm{Lat}\, T = \{(0), \mathcal{F}\}$

** ~~Much~~ more recently[2019] Tcaciuc gave a reformulation of ~~the~~ <u>ISP(T) for T quasinilpotent:</u>

there exists F of rk 1 and $\alpha \notin \{0,1\}$ s.t. $T+F$ and $T+\alpha F$ are also quasinilpotent *

(*** around hypercyclicity)

GRIVAUX ('12)
$\exists\, U$ unitary and R rk 1 such that $U+R$ is hypercyclic

III Extension of Dual Algebra Techniques

|| '78 S. BROWN $S \in \mathcal{L}(H)$ subnormal has i.s.

- powerfull method involving isometric representation of $H^\infty(\mathbb{D})$ in $\mathcal{L}(H)$

- above all additional structure results

  - in particular Olin. Thomson '80 every subnormal op. is reflexive

  - very successful for contractions with isometric Nagy-Foias functional calculus i.e. the class $\mathbb{A}$ and its subclasses $\mathbb{A}_{m,n}$ $1 \leq m, n \leq \aleph_0$ (e.g. the Bergman shift) reflexivity of $T \in \mathbb{A}$, Brown-C 88)

A lot of work has been done with, instead of $\mathbb{D}$, ~~an "arbitrary~~ a more general domain in $\mathbb{C}$ or $\mathbb{C}^n$

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Standard Context

- An isomorphic representation $H^\infty(G) \to \mathcal{L}(\mathcal{X})$
  $h \to h(T)$
- a bilinear map $\square : \mathcal{H} \times \mathcal{H}^* \to$ predual of $H^\infty(G)$
  $x \square y^* : h \to \langle h(T)x, y^* \rangle$

Exercise. if for some x, y^* $x \square y^* = E_\lambda$ (evaluation at $\lambda \in G$ $h \to h(\lambda)$) then we have ("G-") invariant subspace for T

In general when the S. Brown works it yields "$\square$" onto (P'pty(A_1))

'03 Ambrozie-Muller

Thm Let $T \in \mathcal{L}(\mathcal{X})$ be polynomially bdd (i.e. $\exists M \quad \|p(T)\| \leq M \|p\|_\infty \quad p \in \mathbb{C}[X]$) with $\sigma(T) \supset \mathbb{T}$ $(:= \partial \mathbb{D})$; then T^* has i.s.

Comments

1. New even for $\mathcal{X} = H$

2. One of the main innovations: Use of classical interpolation in H^∞ (see book [Ch-P])

3 A year or so later came the associated factorizations and reflexivity results (Rejasse; A-M)

<u>Extension</u>

Thm Let T have $\bar{G}$ as an M-spectral set where G is a bdd finitely connected domain in $\mathbb{C}$; if $\sigma(T) \supset \partial G$ then T^* has a rationally i.s.

Yavuz IEOT 2007

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IV Other directions

1. Universal Operators

Dfn U is universal for $\mathcal{X}$ if $\forall T \in \mathcal{L}(\mathcal{X})$ $\exists \mathcal{M} \in \operatorname{Lat} U$ and $\exists \lambda \neq 0$, s.t. $\lambda T \cong U/\mathcal{M}$

Rota's exple ('60): backward shift of infinite multiplicity

Caradus ('69) $U \in \mathcal{L}(H)$ s.t. $\begin{cases} 1 & \dim \ker U = \infty \\ 2 & \operatorname{Ran} U = H \end{cases}$ is universal

Foias-Pearcy 1978 a model for quasinilpotent (as part of a weighted backward quasi-nilpotent shift of infinite multiplicity)

Circle of ideas revisited in recent work of Cowen-Gallardo

2. Almost invariant subspaces (APTT '09)

dfn $\mathcal{Y} \subset \mathcal{X}$ subspace is almost invariant for $T \in \mathcal{L}(\mathcal{X})$ if there exists $\mathcal{F}$ finite dim'l s.t.

$$T\mathcal{Y} \subset \mathcal{Y} + \mathcal{F}.$$

makes sense only for $\begin{cases} \dim \mathcal{Y} = \infty \\ \operatorname{codim} \mathcal{Y} = \infty \end{cases}$

(otherwise $\mathcal{Y}$ is almost invariant for any $T \in \mathcal{L}(\mathcal{H})$; such a $\mathcal{Y}$ is called half space

Thm (Tcaciuc '17) Let $T \in \mathcal{L}(\mathcal{X})$; then there exists $\mathcal{R}$ of $\dim \leq 1$, and $\mathcal{Y}$ half space s.t. $T\mathcal{Y} \subset \mathcal{Y} + \mathcal{R}$

V "Simple" perturbations of "well-known" operators

Let H with an o.n.b. $(e_n)_n$ and D a bounded diagonal operator $De_n = \lambda_n e_n$ $(\lambda_n)_n \in \ell^\infty$ assumed 1-to-1; let $u = \sum_{n\geq 0} a_n e_n$, $v = \sum_{n\geq 0} b_n e_n$ in H and nonzero

Pb Does $D + u\otimes v$ has its?

This is an old test question

Breakthrough in '07 Ko-Jung-Foias-Pearcy u, v "$\in \ell^{2/3}$"

Recent progress

Fang-Xia '12 same with $\sum_{i=1}^{n} u_i \otimes v_i$ $u_i, v_i \in \ell^1$

Klaja '15 ~~[illegible]~~ compact perturbations instead of $u\otimes v$.

Albrecht-Ch.

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THANK YOU FOR

YOUR ATTENTION