Exponential frames and syndetic Riesz sequences

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$\mathcal{H}$ separable Hilbert space, I a countable set.

Definition

$\{\varphi_i\}_{i\in I} \subset \mathcal{H}$ is a *frame* with bounds $0 < A \leq B < \infty$ if

$$A\,\|f\|^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B\,\|f\|^2$$

for all vectors $f \in \mathcal{H}$. $\{\varphi_i\}_{i\in I}$ is a *Bessel sequence* if $A = 0$.

Definition

$\{\varphi_i\}_{i\in I} \subset \mathcal{H}$ a *Riesz sequence* in $\mathcal{H}$ with bounds $0 < A \leq B < \infty$ if

$$A \sum_{i\in I} |a_i|^2 \leq \left\| \sum_{i\in I} a_i \varphi_i \right\|_{\mathcal{H}}^2 \leq B \sum_{i\in I} |a_i|^2$$

for every finite sequence of scalars $\{a_i\}_{i\in I}$.

Paley-Wiener space

Definition

$\mathcal{S} \subset \mathbb{R}$ set of finite positive Lebesgue measure.
Define Paley-Wiener space

$$PW_{\mathcal{S}} = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \setminus \mathcal{S}\}.$$

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Definition

$\Lambda \subset \mathbb{R}$ a countable set.
$\Lambda \subset \mathbb{R}$ is *sampling set* for $PW_{\mathcal{S}}$ if $\exists A, B > 0$

$$A||f||_2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B||f||^2 \qquad \text{for all } f \in PW_{\mathcal{S}}.$$

$\Lambda \subset \mathbb{R}$ is *interpolation set* for $PW_{\mathcal{S}}$ if for every $c \in \ell^2(\Lambda)$ there exists $f \in PW_{\mathcal{S}}$ such that

$$f(\lambda) = c_\lambda \qquad \text{for all } \lambda \in \Lambda.$$

Dictionary: complex analysis $\leftrightarrow$ frame theory

Define exponential system $E(\Lambda) = \left\{e^{i\lambda x}\right\}_{\lambda\in\Lambda}$.

Theorem

Suppose $\mathcal{S} \subset \mathbb{R}$ is a bounded set and $\Lambda \subset \mathbb{R}$ is uniformly discrete

$$\inf_{\lambda,\mu\in\Lambda,\lambda\neq\mu} |\lambda - \mu| > 0.$$

- *Λ is a sampling set for $PW_{\mathcal{S}} \iff E(\Lambda)$ is a frame in $L^2(\mathcal{S})$.*

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- *Λ is an interpolation set for $PW_{\mathcal{S}}$ $\iff$ $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$.*

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Example

$\mathcal{S} \subset \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \cong [-\pi, \pi)$ set of positive Lebesgue measure.
$\mathbb{Z}$ is a sampling set for $PW_{\mathcal{S}}$.
$E(\mathbb{Z})$ is a tight frame in $L^2(\mathcal{S})$ with bound 2π.

Classical results

Theorem (Kahane (1957))

Let $I \subset \mathbb{R}$ be an interval. If the upper density

$$D^+(\Lambda) := \lim_{r\to\infty} \sup_{a\in\mathbb{R}} \frac{\#(\Lambda \cap (a, a+r))}{r} < \frac{|I|}{2\pi},$$

then $E(\Lambda)$ is a Riesz sequence in $L^2(I)$. On the other hand if $D^+(\Lambda) > \frac{|I|}{2\pi}$ then $E(\Lambda)$ is not a Riesz sequence in $L^2(I)$.

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Theorem (Landau (1967))

Let $\mathcal{S}$ be a measurable set. If $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$ then $D^{+}(\Lambda) \leq \frac{|\mathcal{S}|}{2\pi}$.

Question

Given a set $\mathcal{S}$, does there exist a set Λ of positive density such that the exponential system $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$?

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Theorem (Bourgain-Tzafriri (1987))

Given $\mathcal{S} \subset \mathbb{T}$ of positive measure, there exists a set $\Lambda \subset \mathbb{Z}$ with positive asymptotic density

$$dens(\Lambda) = \lim_{r\to\infty} \frac{\#(\Lambda \cap (-r, r))}{2r} > c\,|\mathcal{S}|$$

and such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. Here c is an absolute constant, independent of $\mathcal{S}$.

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and such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. Here c is an absolute constant, independent of $\mathcal{S}$.

Every space $PW_{\mathcal{S}}$, $\mathcal{S} \subset \mathbb{T}$, has an interpolation set $\Lambda \subset \mathbb{Z}$ with positive upper density proportional to $|\mathcal{S}|$.

Definition

A subset $\Lambda = \{\ldots < \lambda_0 < \lambda_1 < \lambda_2 < \ldots\} \subset \mathbb{Z}$ is *syndetic* if the consecutive gaps in Λ are bounded

$$\gamma(\Lambda) := \sup_{n \in \mathbb{Z}} (\lambda_{n+1} - \lambda_n) < \infty.$$

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Theorem (Lawton (2010) and Paulsen (2011))

Given a set $\mathcal{S} \subset \mathbb{T}$ of positive measure, TFAE:

(i) *There exists $r \in \mathbb{N}$ and a partition $\mathbb{Z} = \bigcup_{j=1}^{r} \Lambda_j$ such that $E(\Lambda_j)$ is a Riesz sequences in $L^2(\mathcal{S})$ for all $j = 1, \ldots, r$.*

(ii) *There exists $d \in \mathbb{N}$ and a syndetic set $\Lambda \subseteq \mathbb{Z}$ with $\gamma(\Lambda) = d$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$.*

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Remark

(ii) $\implies$ (i) can take $r \leq d$ by considering translates of Λ.
(i) $\implies$ (ii) no upper bound on d in terms of r.

Theorem (the Feichtiner conjecture)

Suppose that $\{u_i\}_{i\in I}$ *is a frame in* $\mathcal{H}$ *such that*

$$\inf_{i\in I} \|u_i\|^2 > 0.$$

Then, I *can be partitioned into subsets* $I_1, \ldots, I_r$ *such that every subfamily* $\{u_i\}_{i\in I_j}$, $j = 1, \ldots, r$, *is a Riesz sequence in* $\mathcal{H}$.

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Theorem (Paley-Wiener space)

Every sampling set Λ *in* $PW_{\mathcal{S}}$ *is a finite union of interpolation sets.*

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Solution of Kadison-Singer Problem

Theorem (Marcus-Spielman-Srivastava (2013))

If $\varepsilon > 0$ *and* $v_1, \ldots, v_m$ *are independent random vectors in* $\mathbb{C}^d$ *with finite support. Then,*

$$\mathbb{E}\left[\sum_{i=1}^{m} v_i v_i^*\right] = \mathbf{I} \qquad \text{and} \qquad \mathbb{E}\left[\|v_i\|^2\right] \leq \epsilon \qquad \text{for all } i$$

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$$\implies \mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \le \left(1+\sqrt{\varepsilon}\right)^2\right) > 0.$$

Improvement for support of size 2

Theorem (B.-Casazza-Marcus-Speegle (2019))

If $0 < \epsilon < 1/2$ *and* $v_1, \ldots, v_m$ *are independent random vectors in* $\mathbb{C}^d$ *with support of size* 2*. Then,*

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$$\implies \qquad \mathbb{P}\left(\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \le 1 + 2\sqrt{\epsilon}\sqrt{1-\epsilon}\right) > 0.$$

Quantitative Feichtinger conjecture

Theorem (B.-Casazza-Marcus-Speegle (2019))

Let $\varepsilon > 0$ and suppose that $\{u_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$ with bound 1 that consists of vectors of norms $\|u_i\|^2 \geq \varepsilon$. Then there exists a universal constant $C > 0$, such that I can be partitioned into $r \leq \frac{C}{\varepsilon}$ subsets $I_1, \ldots, I_r$ such that every subfamily $\{u_i\}_{i\in I_j}$, $j = 1, \ldots, r$ is a Riesz sequence in $\mathcal{H}$. Moreover, if $\varepsilon > 3/4$, then $r = 2$ works.

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Corollary

There exists a universal constant $C > 0$ such that for any subset $\mathcal{S} \subset \mathbb{T}$ with positive measure, the exponential system $E(\mathbb{Z})$ can be decomposed as a union of $r \leq \frac{C}{|\mathcal{S}|}$ Riesz sequences $E(\Lambda_j)$ in $L^2(\mathcal{S})$ for $j = 1, \ldots, r$. Moreover, if $\frac{|\mathcal{S}|}{2\pi} > 3/4$, then $r = 2$ works.

Remark

Lawton's Theorem and the solution of Kadison-Singer problem (Feichtinger conjecture) $\Longrightarrow$ there exists a syndetic Riesz sequence of exponentials in $L^2(\mathcal{S})$.

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Question (Olevskii)

What is the bound on a gap $\gamma(\Lambda)$ for syndetic $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$?

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Lawton's Theorem and the solution of Kadison-Singer problem (Feichtinger conjecture) $\implies$ there exists a syndetic Riesz sequence of exponentials in $L^2(\mathcal{S})$.

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Theorem (Nitzan, Olevskii, Ulanovskii (2016))

For every $\mathcal{S} \subset \mathbb{R}$ *of finite measure, the space* $L^2(\mathcal{S})$ *admits an exponential frame.*

Every space $PW_{\mathcal{S}}$, where $|\mathcal{S}| < \infty$, has a sampling set.

Main result

Theorem (B.-Londner (2019))

Let $\varepsilon > 0$ and suppose that $\{u_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$ with bound 1 and

$$\|u_i\|^2 \geq \varepsilon \qquad \forall i \in I.$$

Suppose $\{J_k\}_k$ is a collection of disjoint subsets of I with

$$\#J_k \geq r = \left\lceil \frac{C}{\varepsilon} \right\rceil \qquad \text{for all } k.$$

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Suppose $\{J_k\}_k$ is a collection of disjoint subsets of I with

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Then there exists a selector, i.e. a subset $J \subset \bigcup_k J_k$ satisfying

$$\#(J \cap J_k) = 1 \qquad \forall k$$

such that $\{u_i\}_{i \in J}$ is a Riesz sequence in $\mathcal{H}$.
Moreover, if $\varepsilon > \frac{3}{4}$, then $r = 2$ works.

Syndetic Riesz sequences

Applying the main theorem to the exponential system $E(\mathbb{Z}) = \{1_{\mathcal{S}} e^{int}\}_{n\in\mathbb{Z}}$ with $J_k = [kr, (k+1)r) \cap \mathbb{Z}$, $k \in \mathbb{Z}$, yields:

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Corollary

There exists a universal constant $C > 0$ such that for any subset $\mathcal{S} \subset \mathbb{T}$ with positive measure, there exists a syndetic set $\Lambda \subset \mathbb{Z}$ with gaps

$$\gamma(\Lambda) \leq C\,|\mathcal{S}|^{-1}$$

so that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. Moreover, if $\frac{|\mathcal{S}|}{2\pi} > \frac{3}{4}$ then such Λ exists with $\gamma(\Lambda) \leq 3$.

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so that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$. Moreover, if $\frac{|\mathcal{S}|}{2\pi} > \frac{3}{4}$ then such Λ exists with $\gamma(\Lambda) \leq 3$.

Every space $PW_{\mathcal{S}}$, $\mathcal{S} \subset \mathbb{T}$, has a syndetic interpolation set $\Lambda \subset \mathbb{Z}$ with gaps proportional to $|\mathcal{S}|^{-1}$.

Higher dimensional syndetic sets 1

Corollary

Let $\mathcal{S} \subset \mathbb{T}^d$ *be any subset of positive measure.*
Let $\mathcal{R} \subset \mathbb{Z}^d$ *be any d-dimensional rectangle with* $\# |\mathcal{R}| > C |\mathcal{S}|^{-1}$*.*
Let $\mathbb{Z}^d = \bigcup \mathcal{R}_k$ *be and any partition into translated copies of* $\mathcal{R}$*.*

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Let $\mathbb{Z}^d = \bigcup \mathcal{R}_k$ *be and any partition into translated copies of* $\mathcal{R}$.
Then there exists a set $\Lambda \subset \mathbb{Z}^d$ *such that*

$$\#|\Lambda \cap \mathcal{R}_k| = 1 \qquad \forall k$$

and $E(\Lambda)$ *is a Riesz sequence in* $L^2(\mathcal{S})$.

Corollary

Let $\mathcal{S} \subset \mathbb{T}^d$ be any subset of positive measure.
Let $\mathcal{R} \subset \mathbb{Z}^d$ be any d-dimensional rectangle with $\# |\mathcal{R}| > C |\mathcal{S}|^{-1}$.
Let $\mathbb{Z}^d = \bigcup \mathcal{R}_k$ be and any partition into translated copies of $\mathcal{R}$.
Then there exists a set $\Lambda \subset \mathbb{Z}^d$ such that

$$\#|\Lambda \cap \mathcal{R}_k| = 1 \qquad \forall k$$

and $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$.
In particular, if $\mathcal{R}$ is a cube, then

$$\sup_{\lambda \in \Lambda} \inf_{\mu \in \Lambda \setminus \{\lambda\}} |\lambda - \mu| \leq C\sqrt{d}\, |\mathcal{S}|^{-\frac{1}{d}}.$$

Higher dimensional syndetic sets 2

Partitioning the lattice $\mathbb{Z}^d$ into thin and long rectangles in a checkerboard way yields:

Corollary

For any subset $\mathcal{S} \subset \mathbb{T}^d$ of positive measure, there exists $\Lambda \subset \mathbb{Z}^d$ so that $E(\Lambda)$ is a Riesz sequence in $L^2(\mathcal{S})$ and every one dimensional section of Λ in every direction

$$\Lambda(k_1, \ldots, \hat{k}_j, \ldots, k_{d-1}) = \{k_j \in \mathbb{Z} : (k_1, \ldots, k_j, \ldots, k_{d-1}) \in \Lambda\}$$

is syndetic for any $(k_1, \ldots, \hat{k}_j, \ldots, k_d) \in \mathbb{Z}^{d-1}$ and $j = 1, \ldots, d$ with gap

$$\gamma(\Lambda(k_1, \ldots, k_{d-1})) \leq Cd\,|\mathcal{S}|^{-1}.$$

Theorem

Let $r, M \in \mathbb{N}$ and $\delta > 0$. Suppose that $\{u_i\}_{i=1}^{M} \subset \mathcal{H}$ is a Bessel sequence with bound 1 and $\|u_i\|^2 \leq \delta$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\# J_k \geq r$ for all k, there exists a subset $J \subset [M]$ such that $\#(J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Bessel sequence with bound

$$\left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2 .$$

Proof.

WLOG $\#J_k = r$. Define independent random vectors v_k: for $k = 1, \dots, n$ the vector v_k takes values $\sqrt{r} u_i$ for any $i \in J_k$ with probability $\frac{1}{r}$. Then,

$$\sum_{k=1}^{n} \mathbb{E}\left(v_k v_k^*\right) \leq \mathrm{I}_{\mathcal{H}} \qquad \text{and} \qquad \mathbb{E}\left\|v_k\right\|^2 \leq r\delta \qquad \forall k.$$

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By Theorem of Marcus-Spielman-Srivastava

$$\mathbb{P}\left(\left\|\sum_{k=1}^{n} v_k v_k^*\right\| \le \left(1 + \sqrt{r\delta}\right)^2\right) > 0.$$

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WLOG $\#J_k = r$. Define independent random vectors v_k: for $k = 1, \ldots, n$ the vector v_k takes values $\sqrt{r} u_i$ for any $i \in J_k$ with probability $\frac{1}{r}$. Then,

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$$\mathbb{P}\left(\left\|\sum_{k=1}^{n} v_k v_k^*\right\| \leq \left(1 + \sqrt{r\delta}\right)^2\right) > 0.$$

which implies the existence of a set $J \subset [M]$ such that

$$\left\|\sum_{i \in J} u_i u_i^*\right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2.$$

□

Selector theorem for short vectors

Theorem

Let $M \in \mathbb{N}$ and $\delta_0 \in \left(0, \frac{1}{4}\right)$. Suppose that $\{u_i\}_{i=1}^{M} \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \le \delta_0$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\#J_k = 2$ for all k, there exists a subset $J \subset [M]$ such that $\#\left(J \cap J_k\right) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Bessel sequence with bound $1 - \varepsilon_0$, where $\varepsilon_0 = \frac{1}{2} - \sqrt{2\delta_0\left(1 - 2\delta_0\right)}$.

Lemma (B.-Casazza-Marcus-Speegle (2019))

Let $P : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto a closed subspace $H \subset \mathcal{H}$, and let $\{e_i\}_{i \in I}$ be an orthogonal basis for $\mathcal{H}$. Then for any subset $J \subset I$ and $\delta > 0$ the following are equivalent:

1. $\{Pe_i\}_{i \in J}$ *is a Bessel sequence with bound* $1 - \delta$.
2. $\{(\boldsymbol{I}_{\mathcal{H}} - P)\, e_i\}_{i \in J}$ *is a Riesz sequence with lower bound* δ.

Naimark's complements

Lemma (B.-Casazza-Marcus-Speegle (2019))

Let $P : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto a closed subspace $H \subset \mathcal{H}$, and let $\{e_i\}_{i\in I}$ be an orthogonal basis for $\mathcal{H}$. Then for any subset $J \subset I$ and $\delta > 0$ the following are equivalent:

1. $\{Pe_i\}_{i\in J}$ *is a Bessel sequence with bound* $1-\delta$.
2. $\{(\boldsymbol{I}_{\mathcal{H}} - P)\, e_i\}_{i\in J}$ *is a Riesz sequence with lower bound* δ.

Corollary

Let $M \in \mathbb{N}$ and $\delta_0 \in \left(0, \frac{1}{4}\right)$. Suppose that $\{u_i\}_{i=1}^{M} \subset \mathcal{H}$ is a Bessel sequence with Bessel bound B and $\|u_i\|^2 \geq B\,(1-\delta_0)$ for all i. Then for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\# J_k = 2$ for all k, there exists a subset $J \subset [M]$ such that $\#\,(J \cap J_k) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i\in J}$ is a Riesz sequence with lower Riesz bound $B\varepsilon_0$.

Combine two selectors theorems with

Lemma

Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $M \in \mathbb{N}$ and $\delta \in (0,1)$. Suppose $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \geq \delta$ for all i. Then for every large enough $K \in \mathbb{N}$, there exist vectors $\varphi_1, \ldots, \varphi_K \in \mathcal{H}$ with $\|\varphi_i\|^2 \geq \delta$ for all i such that $\{u_i\}_{i=1}^M \cup \{\varphi_i\}_{i=1}^K$ is a Parseval frame for its linear span.

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Theorem (Finite version of main result)

Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Suppose that $\{u_i\}_{i=1}^M \subset \mathcal{H}$ is a Bessel sequence with Bessel bound 1 and $\|u_i\|^2 \geq \varepsilon$ for all i. Then there exists $r = O\left(\frac{1}{\varepsilon}\right)$, independent of M, such that for every collection of disjoint subsets $J_1, \ldots, J_n \subset [M]$ with $\#J_k \geq r$ for all k, there exists a subset $J \subset [M]$ such that $\#\left(J \cap J_k\right) = 1$ for all $k \in [n]$ and the system of vectors $\{u_i\}_{i \in J}$ is a Riesz sequence with lower Riesz bound $\varepsilon\varepsilon_0$. Moreover, if $\varepsilon > \frac{3}{4}$ then the same conclusion holds with $r = 2$.

Diagonal argument with the pigeonhole principle

Lemma

Let $\{J_k\}_k$ be a collection of disjoint subsets of I. Assume for every $n \in \mathbb{N}$ we have a subset $I_n \subset \bigcup_{k=1}^n J_k$ such that

$$\#(I_n \cap J_k) = 1 \quad \text{for } k = 1, \ldots, n$$

Then, there exists a subset $I_\infty \subset I$ and an increasing sequence $\{n_j\}$ such that

$$I_{n_j} \cap \left(\bigcup_{k=1}^{j} J_k \right) = I_\infty \cap \left(\bigcup_{k=1}^{j} J_k \right)$$

In particular, we have

$$\#(I_\infty \cap J_k) = 1 \qquad \forall k.$$

This yields infinite dimensional version of main result.

Theorem (R_ε conjecture of Casazza-Tremain)

Let $\{u_i\}_{i\in I}$ be a unit norm Bessel sequence in $\mathcal{H}$ with bound B. Then there exists a universal constant $C > 0$ such that for any $\varepsilon > 0$ and any collection of disjoint subsets of I, $\{J_k\}_k$ satisfying $\#J_k \geq r = \lceil C\frac{B}{\varepsilon^4}\rceil$, for all k. There exists a selector $J \subset \bigcup_k J_k$ satisfying

$$\#(J \cap J_k) = 1 \qquad \forall k$$

and such that $\{u_i\}_{i\in J}$ is a Riesz sequence in $\mathcal{H}$ with bounds $1 \pm \varepsilon$.

Syndetic sets and almost tight Riesz bounds

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Remark

A multi-paving result of Ravichandran-Srivastava (2017) suggest that $r=O(\frac{B}{\varepsilon^2})$ should work. Hence, this would yield syndetic Riesz sequences of exponentials in $L^2(\mathcal{S})$ with Riesz bounds $\frac{|\mathcal{S}|}{2\pi}(1\pm\varepsilon)$ and gaps $O(\frac{1}{|\mathcal{S}|\varepsilon^2})$ instead of $O(\frac{1}{|\mathcal{S}|\varepsilon^4})$.

Open problems on syndetic Riesz sequences

Question

Does an arbitrary measurable set $\mathcal{S} \subset \mathbb{T}$ of (almost full) measure admit an exponential Riesz sequence $E(\Lambda)$, $\Lambda \subset \mathbb{Z}$, such that

$$\inf_{\lambda,\mu \in \mathbb{Z}\setminus\Lambda, \lambda \neq \mu} |\lambda - \mu| \geq \frac{C}{|\mathbb{T} \setminus \mathcal{S}|}?$$

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Question (Olevskii)

Does an arbitrary measurable set $\mathcal{S} \subset \mathbb{T}$ of finite measure admit an exponential Riesz **basis** *$E(\Lambda)$ for some $\Lambda \subset \mathbb{R}$?*

THANK YOU FOR ATTENTION