Limits of harmonic and holomorphic functions along segments ending at the boundary

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Interpolation and Sampling

Fields Institute, 14 September 2021

$S \subset X$

Extending a function $f : S \to Y$

to

a function $F : X \to Y$

Interpolation: existence.

Sampling: construction - use f to describe and find F.

$$S \subset X, \quad f : S \to Y$$

Extension

$$F : X \to Y, \quad |F - f| = 0$$

Approximation

$$F : X \to Y, \quad |F - f| < \epsilon$$

Analysis: extensions unique, “so" usually do not exist.

Analytic agnostic: believes there is at most one God.

Approximation: analyst’s substitute for extension.

Approximation: a GOOD substitute for extension

$$S \subset X, \quad f : S \to Y$$

Extension: $\exists F : X \to Y$,

$$|F - f| = 0$$

Approximation: $\forall \epsilon > 0$, $\exists F : X \to Y$,

$$F : X \to Y, \quad |F - f| < \epsilon$$

Heisenberg Uncertainty Principle: If ϵ small enough (smaller than diameter of an electron ?), no future instrument will detect difference

between ϵ and 0,
hence
between F and f.

Complex approximation

On compact sets $K \subset \mathbb{C}$ well known.

On closed sets $F \subset \mathbb{C}$, well developed, not well known.

Carleman 1927. For arbitrary continuous functions φ and $\epsilon > 0$, on $\mathbb{R}$, there exists an entire function f, such that

$$|f(x) - \varphi(x)| < \epsilon(x), \quad \forall x \in \mathbb{R}.$$

Dirichlet problem for upper half-plane. (Nevanlinna 1925) Given φ continuous on $\mathbb{R}$, there exists u continuous on $y \geq 0$, harmonic on $y > 0$,

$$u(x) = \varphi(x), \quad \forall x \in \mathbb{R}.$$

Proof (Kaplan 1955). Carleman $\Rightarrow$ $\exists f$ entire.

$$|f - \varphi| < 1.$$

Solution

$$u = P_{\varphi - f} + f,$$

where $P_{\varphi - f}$ is the Poisson integral of $\varphi - f$ for the upper half-plane.

Negilgeable sets in potential theory

A set E is said to be polar, if there is a subharmonic function s in a neighbourhood of E, such that $E \subset s^{-1}(-\infty)$.

A compact set is polar iff it is of zero capacity.

Polar sets are undetectable by Brownian motion.
That is, Brownian motion almost surely misses them.

Polar sets are removeable singularities for bounded harmonic functions.

Cave domain $U \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, $n > 2$

$U = (U', c, d) =$
$\{(x', y) \in \mathbb{R}^{n-1} \times \mathbb{R} : x' \in U',\ c(x') < y < d(x')\}$

Region of $\mathbb{R}^n$ between the graphs of two functions c and d defined on a domain $U' \subset \mathbb{R}^{n-1}$

Theorem. Let $U = (U', c, d)$ be a cave domain and $F' \subset U'$ an F_σ polar set. Then, for every $\varphi \in C(U)$, there is a harmonic function h on U such that $\forall x' \in F'$,

$$(h - \varphi)(x', y) \to 0, \quad \text{as} \quad y \downarrow c(x'), \quad \text{or} \quad y \uparrow d(x').$$

d ———————— $(x', d(x'))$ ————————————

$h \to$

$y \quad U \qquad \cdot \quad (x', y)$

$\varphi \to$

c ———————— $(x', c(x'))$ ————————————

Example. Countable dense subsets of $\mathbb{R}^{n-1}$ are polar.
Main example. U could be upper half-space $\mathbb{R}^{n-1} \times \mathbb{R}^+$.

Starlike domain $U \subset \mathbb{R}^n$. In polar coordinates,
$U = (U', R) =$
$\{(\theta, \rho) \in \mathbb{R}^n = S^{n-1} \times [0, +\infty), : \theta \in U', \rho < R(\theta)\}$.

Starlike shell domain $U \subset \mathbb{R}^n$.
$U = (U', r, R) =$
$\{(\theta, \rho) \in S^{n-1} \times (0, +\infty), : \theta \in U', \ r(\theta) < \rho < R(\theta)\}$.

We consider a starlike domain to be a degenerate starlike shell domain.

Theorem. Let $U \subset \mathbb{R}^n$ be a starlike shell domain and $F' \subset U'$ an F_σ polar set. Then, for every $\varphi \in C(U)$, there is a harmonic function h on U such that, $\forall \theta \in F'$,

$$(h - \varphi)(\theta, \rho) \to 0, \quad \text{as} \quad \rho \downarrow r(\theta), \quad \text{or} \quad \rho \uparrow R(\theta).$$

Complex analysis

Lehto 1955.

If $\psi_1, \psi_2 : [0, 2\pi) \to [-\infty, +\infty]$ are measurable functions, there exists f holomorphic in $\mathbb{D} \subset \mathbb{C}$, such that

$$\lim_{r \nearrow 1} f(re^{i\theta}) = \psi_1(\theta) + i\psi_2(\theta), \quad for \quad a.e. \quad \theta \in [0, 2\pi).$$

Theorem.

If $\psi_1, \psi_2 : \mathbb{T}^n \to [-\infty, +\infty]$ are measurable functions, there exists $f = u + iv$ holomorphic in $\mathbb{D}^n$, such that

$$\lim_{r \nearrow 1} u(r\theta) = \psi_1(\theta), \quad \lim_{r \nearrow 1} v(r\theta) = \psi_2(\theta), \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Remark. This does not follow from potential theory. Potential theory might give us harmonic functions u and v, but we would need for u and v to satisfy the system of Cauchy-Riemann equations.

Approximating continuous functions

For a compact set $K \subset \mathbb{R}^n$, denote by $P(K)$ the functions on K which are uniform limits of polynomials.

Weierstrass 1885.
If I is a closed interval, then $P(I) = C(I)$.

Stone-Weierstrass. A subalgebra $A \subset C(K)$ is dense in $C(K)$ if it contains constants and separates points.

Corollary. If $K \subset \mathbb{R}^n$, then $P(K) = C(K)$.

Complex approximation

For a compact set $K \subset \mathbb{C}^n$, denote by $C(K)$ the continuous (complex) functions on K and by $P(K)$ the functions on K which are uniform limits of (complex) polynomials.

Complex Stone-Weierstrass. A subalgebra $A \subset C(K)$ is dense in $C(K)$ if it contains constants, separates points and contains conjugates.

This is not a theorem on complex analysis, but merely a theorem on complex algebra and complex topology, because the conjugate of a (complex) analytic function is rarely a (complex) analytic function. For example, the conjugate of the function $z \mapsto z$ is the function $z \mapsto \bar{z}$, which is not (complex) analytic.

Authentic complex approximation

Walsh 1926. If $J \subset \mathbb{C}$ is a Jordan arc, then $P(J) = C(J)$.

Lavrentiev 1936.
If $K^\circ = \emptyset$, $P(K) = C(K)$ iff $\mathbb{C} \setminus K$ is connected.

For approximation on closed sets $F \subset \mathbb{C}$, it is natural to approximate by entire functions rather than polynomials.

Definition. $F \subset \mathbb{C}$ is a Carleman set if, for arbitrary continuous functions φ and $\varepsilon > 0$ on F, there exists an entire function f, such that

$$|f(z) - \varphi(z)| < \varepsilon(z), \quad \forall z \in F.$$

Carleman 1927. $\mathbb{R}$ is a Carleman set.

Arakelian 1964. If $F^\circ = \emptyset$, then F is a Carleman set iff $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected and locally connected.

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Arakelian's original formulation. If $F^\circ = \emptyset$, then F is a Carleman set iff, for each $r > 0$, there exists $r' > r$, such that each point in $\mathbb{C} \setminus F$ ouside $|z| = r'$ can be connected to ∞ by a path in $\mathbb{C} \setminus F$ outside $|z| = r$.

Example.

$$F = \left\{ z = x + iy : x \neq 0,\ y = \frac{1}{x} \sin\left(\frac{1}{x}\right) \right\} \cup i\mathbb{R}.$$

$(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected but not locally connected.

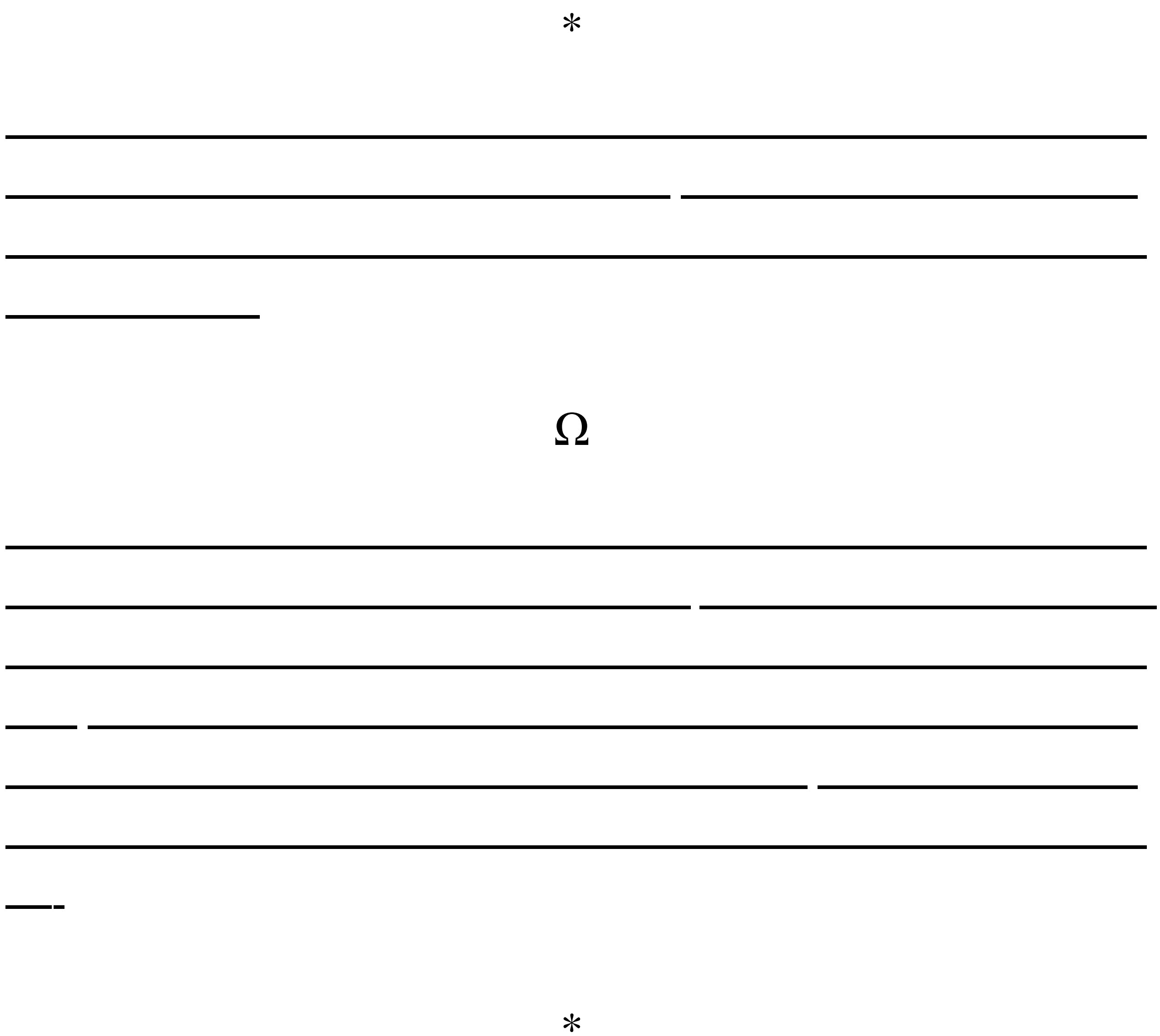
Generalization. $\Omega \subset \mathbb{C}$ open. $F \subset \Omega$ closed. F is a Carleman set in Ω if, for every φ and ε in $C(F)$, with $\varepsilon > 0$, there exists f holomorphic in Ω such that

$$|f(z) - \varphi(z)| < \varepsilon(z), \quad \forall z \in F.$$

Arakelian 1968. $F^\circ = \emptyset$. $F \subset \Omega$ is a Carleman set in Ω iff $\Omega^* \setminus F$ is connected and locally connected, where $\Omega^* = \Omega \cup \{*\}$ is the one-point compactification of Ω.

Arakelian 1968. $F^\circ = \emptyset$. $F \subset \Omega$ is a Carleman set in Ω iff $\Omega^* \setminus F$ is connected and locally connected.

Example: Ω a rectangle and F a union of horizontal lines. Then, $\Omega^* \setminus F$ is connected and locally connected.



Arakelian 1968. $F^\circ = \emptyset$. $F \subset \Omega$ is a Carleman set in Ω iff $\Omega^* \setminus F$ is connected and locally connected.

Example: Ω a disc of finite or infinite radius and F a union of rays. Then, $\Omega^* \setminus F$ is connected and locally connected.

Arakelian 1964. If $F^\circ = \emptyset$, then F is a Carleman set iff $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected and locally connected.

Boivin thesis 1984 $F^\circ = \emptyset$ in open Riemann surface Ω. Then F is a Carleman set iff $\Omega^* \setminus F$ is connected and locally connected.

Reminder

Walsh 1926. If $J \subset \mathbb{C}$ is a Jordan arc, then $P(J) = C(J)$.

Not true in $\mathbb{C}^n$, $n > 1$.

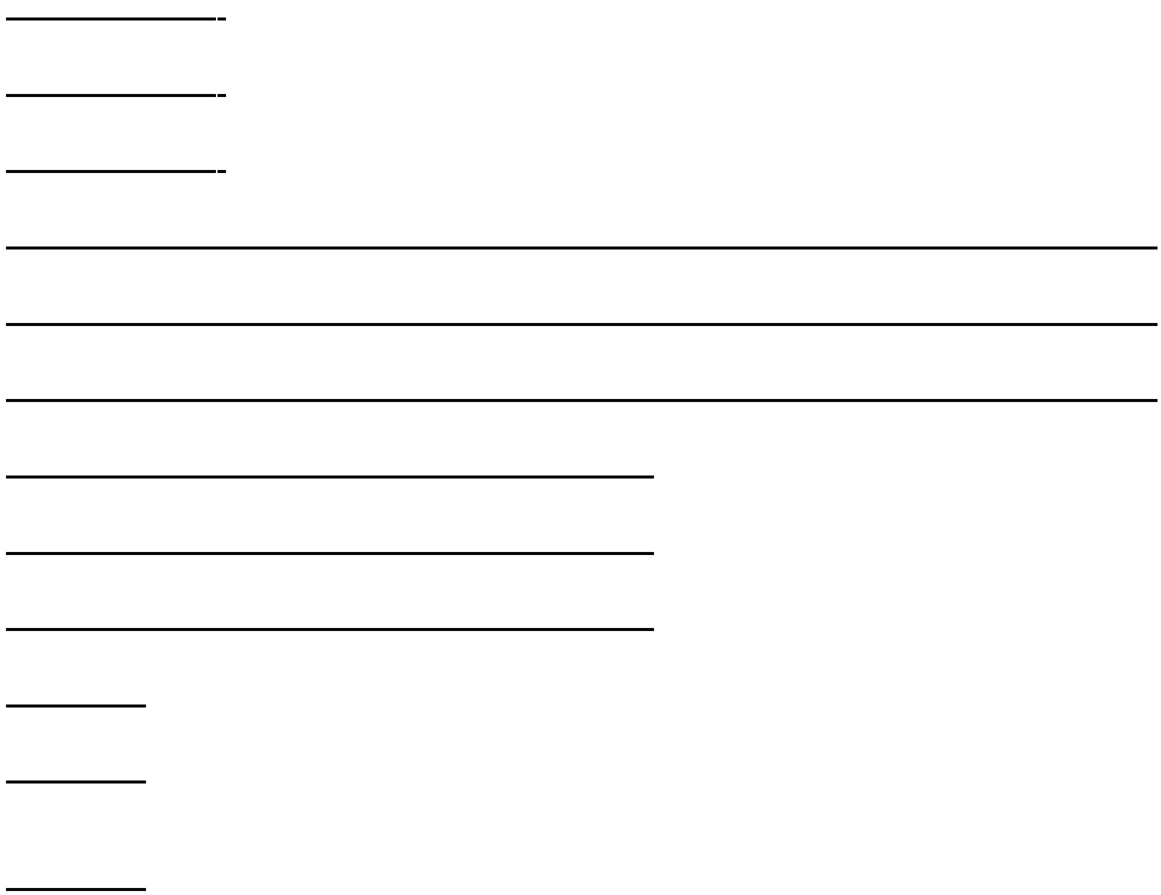
Chacrone, G, Nersessian 1998. Ω_j open Riemann surfaces, $j = 1, \ldots, n$. F_j Carleman sets in Ω_j. Then, $F = F_1 \times \cdots \times F_n$ is a Carleman set in $\Omega = \Omega_1 \times \cdots \times \Omega_n$.

Theorem.
If $\varphi \in C(\mathbb{D}^n)$, there is a function f holomorphic in $\mathbb{D}^n$, such that

$$(f - \varphi)(r\theta) \to 0, \quad \text{as} \quad r \nearrow 1, \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Proof for the disc $\mathbb{D}^1$.

Pretend this is the disc and the left edge is the unit circle and horizontal lines are rays.



$\exists F'_j \subset \mathbb{T}$, closed nowhere dense. $F' = \cup_j F'_j$, $m(F') = 2\pi$.
$F_j = \{z = re^{i\theta} : r_j \leq r < 1,\ e^{i\theta} \in F'_j\}, \quad r_j \nearrow 1$

$F = F_1 \cup F_2 \cup \cdots$

$F = F_1 \cup F_2 \cup \cdots$

Each F_j is a closed nowhere dense set of rays each of which is of length $1 - r_{,}$. which tends to zero. The collection of F_j's is a locally finite family, so the union F is closed nowhere dense. $\mathbb{D}^* \setminus F$ is connected and locally connected so F is a Carleman set in $\mathbb{D}$. We can approximate a continuous function φ on F by functions holomorphic on $\mathbb{D}$.

Chacrone, G, Nersessian 1998. Ω_j open Riemann surfaces, $j = 1, \ldots, n$. F_j Carleman sets in Ω_j. Then,
$F = F_1 \times \cdots \times F_n$ is a Carleman set in $\Omega = \Omega_1 \times \cdots \times \Omega_n$.

Theorem.
If $\varphi \in C(\mathbb{D}^n)$, there is a function f holomorphic in $\mathbb{D}^n$, such that

$$(f - \varphi)(r\theta) \to 0, \quad \text{as} \quad r \nearrow 1, \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Proof. Case $n = 1$ and CGN Theorem.

We have just proved:

Theorem 1.
If $\varphi \in C(\mathbb{D}^n)$, there is a function f holomorphic in $\mathbb{D}^n$, such that

$$(f-\varphi)(r\theta) \to 0, \quad \text{as} \quad r \nearrow 1, \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Theorem 2.
If $\psi_1, \psi_2 : \mathbb{T}^n \to [-\infty, +\infty]$ are measurable functions, there exists $f = u + iv$ holomorphic in $\mathbb{D}^n$, such that

$$\lim_{r \nearrow 1} u(r\theta) = \psi_1(\theta), \quad \lim_{r \nearrow 1} v(r\theta) = \psi_2(\theta), \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Theorem 2 follows from Theorem 1, if we can prove the following.

Lemma. Set $\psi = \psi_1 + i\psi_2$. Then, there exists $\varphi \in C(\mathbb{D}^n)$ having radial limits $\psi(\theta)$, at almost all $\theta \in \mathbb{T}^n$.

May seem contradictory, but we are not asking that φ have a continuous extension to even one single point of $\mathbb{T}^n$.

Lemma

Fix $n \in \mathbb{N}$, $0 < c \leq +\infty$,
measurable $\psi_1, \psi_2 : \mathbb{T}^n \to [-c, +c]$, $\quad \psi = \psi_1 + i\psi_2$.
Then, there exists $\varphi = \varphi_1 + i\varphi_2 \in C(\mathbb{D}^n)$:

$$\lim_{r \nearrow 1} \varphi(r\theta) = \psi(\theta) \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Proof. . May assume $c = 1$. By Lusin's theorem,
$\exists\, F'_j \subset \mathbb{T}^n$ disjoint compact nowhere dense, $j = 1, 2, \ldots$.
The restriction of ψ to each F'_j is continuous.
For $F' = F'_1 \cup F'_2 \cup \cdots$, Haar measure $m(F') = 1$.
Set $F_j = \{r\theta : (1 - 1/j) \leq r < 1,\ \theta \in F'_j\}$.
Extend the restriction of ψ on F'_j to a continuous function
$\varphi : F'_j \cup F_j \to [-1, +1], \quad \varphi : F_j \to (-1, +1), \quad \varphi|_{F'_j} = \psi$

$$\lim_{r \nearrow 1} \varphi(r\theta) = \psi(\theta), \qquad \text{for all} \quad \theta \in F'_j, \quad j = 1, 2, \ldots.$$

$(F_j)_j$ locally finite family closed sets, so $F = \cup_j F_j$ closed and $\varphi : F \to (-1, +1)$ continuous. Extend φ continuously $\varphi : \mathbb{D}^n \to (-1, +1)$. $\square$

Recall: We just now proved

Lemma

$\forall$ measurable $\psi_1, \psi_2 : \mathbb{T}^n \to [-\infty, +\infty]$, $\quad \psi = \psi_1 + i\psi_2$.

$\exists$ continuous $\varphi_1, \varphi : \mathbb{D}^n \to (-\infty, +\infty)$, $\quad \varphi = \varphi_1 + i\varphi_2$.

$$\lim_{r \nearrow} \varphi(r\theta) = \psi(\theta) \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Theorem

Fix $\varphi \in C(\mathbb{D}^n)$,

There exists function f holomorphic in $\mathbb{D}^n$, such that

$$(f - \varphi)(r\theta) \to 0, \quad \text{as} \quad r \nearrow 1, \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n.$$

Theorem

If $\psi_1, \psi_2 : \mathbb{T}^n \to [-\infty, +\infty]$ are measurable functions, there exists $f = u + iv$ holomorphic in $\mathbb{D}^n$, such that

$$\lim_{r \nearrow 1} u(r\theta) = \psi_1(\theta), \quad \lim_{r \nearrow 1} v(r\theta) = \psi_2(\theta), \quad a.e. \quad \theta \in \mathbb{T}^n.$$

The set of boundary points at which a continuous function has radial limits is measurable and the radial limit function is measurable on this measurable set. We have the following converse.

Theorem.
$\forall$ measurable $A^1, A^2 \subset \mathbb{T}^n$
$\forall$ measurable $\psi_1 : A^1 \to [-\infty, +\infty]$, $\psi_2 : A^2 \to [-\infty, +\infty]$,
$\exists$ holomorphic $f : \mathbb{D}^n \to \mathbb{C}$, $\quad f = u + iv$, such that

$$\lim_{r \nearrow 1} u(r\theta) = \psi_1(\theta), \quad \text{for} \quad a.e. \quad \theta \in A^1;$$

$$\lim_{r \nearrow 1} v(r\theta) = \psi_2(\theta), \quad \text{for} \quad a.e. \quad \theta \in A^2;$$

$$C_R(u, \theta) = [-\infty, +\infty], \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n \setminus A^1;$$

$$C_R(v, \theta) = [-\infty, +\infty], \quad \text{for} \quad a.e. \quad \theta \in \mathbb{T}^n \setminus A^2,$$

where $C_R(u, \theta)$ is the radial cluster set of u at θ (similarly for v).

$$C_R(u, \theta) = \{w \in [-\infty, +\infty] : \exists t_k \nearrow 1,\ lim_{k\to\infty} u(t_k\theta) = w\}$$

$C_R(u, \theta)$ is the set of values approached radially
by u at the boundary point θ.

What we have shown:

For harmonic functions, we can prescribe limits along segments ending at a subset of the boundary which is small in the sense of potential theory, a countable union of closed polar sets.

For holomorphic functions on the polydisc, we can prescribe limits along radii ending at a subset of the boundary which is topologically small, a countable union of closed nowhere dense sets.

An important difference is that sets which are topologically small can have full measure, while polar sets are of measure zero.

Krantz and Min (2020) have a result for holomorphic functions on the ball in $\mathbb{C}^n$, which is similar to our result for the polydisc. Their work extends work by Hakim and Sibony.

For holomorphic approximation, we used conditions which are both necessary and sufficient, in order for a set to be a Carleman set.

For harmonic approximation, these same conditions are no longer necessary, but they are still sufficient, provided we add a condition on polarity.

Thus, the only difficulty in translating the holomorphic proof to the harmonic proof resides in properties of polar sets. An essential tool is a result of Fuglede in axiomatic potential theory regarding the relations between projections and liftings of polar sets in harmonic spaces.

MERCI !