# Marcinkiewicz-Zygmund Families for Polynomials in Bergman, Hardy, and Fock Spaces


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# 1. Marcinkiewicz-Zygmund families

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## Abstract Problem

$\mathcal{H} \subseteq L^2(X, \mu)$ reproducing kernel Hilbert space with kernel $k$:

$$f(x) = \langle f, k_x \rangle = \int_X f(y) \overline{k_x(y)} \, d\mu(y) = \int_X f(y) k(x, y) \, d\mu(y)$$

Goal [1]: investigate, construct sampling sets for $\mathcal{H}$, i.e.,
$\Lambda \subseteq X$ is a *sampling set* for $\mathcal{H}$, if $\exists A, B > 0$, such that

$$A \|f\|_2^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le B \|f|_2^2 \qquad \forall f \in \mathcal{H}$$

[1] $k_x(y) = \overline{k(x, y)} = k(y, x)$

## Simpler (?) Goal

Study finite-dimensional approximations to sampling sets.

- Let

$$\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \dots \subseteq \mathcal{P}_n \subseteq \mathcal{P}_{n+1} \subseteq \dots$$

be a sequence of finite-dimensional subspaces of $\mathcal{H}$ with *intrinsic* reproducing kernel $k_n$, i.e.,

$$p(x) = \langle p, k_{n,x} \rangle \qquad \forall p \in \mathcal{P}_n$$

and $k_{n,x}(y) = k_n(y, x)$

- $k_n(y, x)$ is kernel of orthogonal projection from $\mathcal{H}$ onto $\mathcal{P}_n$ and $k_{n,x} = P_n k_x$.

- Hope: perhaps finite-dimensional sampling problem in $\mathcal{P}_n$ is easier (is it?)

## Marcinkiewicz-Zygmund families

Sampling sets: $\Lambda_n = \{\lambda_{n,k} : k = 1, \dots, L_n\} \subseteq X$.

**Definition**

An array $\{\Lambda_n\}$ is called Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|p\|_2^2 \leq \sum_{\lambda \in \Lambda_n} \frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} \leq B\|p\|_2^2 \qquad \textit{for all } p \in \mathcal{P}_n \,.$$

Why normalization with $k_n(\lambda,\lambda)$?

- Intrinsic to subspace $\mathcal{P}_n$ (Shapiro, Shields)
- $p(x) = \langle p, k_{n,x}\rangle$ and $\|k_{n,x}\|_2^2 = k_n(x,x)$ therefore

$\left\| \frac{k_{n,x}}{\|k_{n,x}\|} \right\|_2 = 1$. Sampling inequality for $\mathcal{P}_n \quad \Leftrightarrow \quad$ frame inequality with norm-1-frame.

# Polynomials

Basic situation:

- $X \subseteq \mathbb{R}$ or $\mathbb{C}$
- $\mathcal{P}_n$ polynomials of degree $n$ in $L^2(X, \mu)$
- For $X \subseteq \mathbb{C} \quad \Rightarrow \quad \overline{\bigcup_{n=0}^{\infty} \mathcal{P}_n}$ is space of analytic functions:

Bergman space $A^2(X)$.

# 2. Bergman Space on $\mathbb{D}$

## Bergman space on the disc

$A^2(\mathbb{D})$ analytic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with norm

$$\|f\|_{A^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(x+iy)|^2 \, dxdy \, .$$

- $\{\sqrt{k+1} z^k : k = 0, 1, \dots\}$ is orthonormal basis
- reproducing kernel for $\mathcal{P}_n$ is

$$k_n(z,w) = \sum_{k=0}^{n} (k+1)(z\bar{w})^k = \frac{1 + (n+1)(z\bar{w})^{n+2} - (n+2)(z\bar{w})^{n+1}}{(1 - z\bar{w})^2} \, .$$

- kernel for $A^2(\mathbb{D})$ is

$$k(z,w) = \lim_{n\to\infty} k_n(z,w) = \frac{1}{(1-z\bar{w})^2} \qquad z, w \in \mathbb{D} \, .$$

## Bergman space: from sampling to Marcinkiewicz-Zygmund families

$$B_{1-\gamma/n} = \{z \in \mathbb{D} : |z| < 1 - \tfrac{\gamma}{n}\} = B(0, 1 - \tfrac{\gamma}{n})$$

**Theorem**

*Assume that $\Lambda \subseteq \mathbb{D}$ is a sampling set for $A^2(\mathbb{D})$. Then for $\gamma > 0$ small enough, the sets*

$$\Lambda_n = \Lambda \cap B_{1-\gamma/n} = \{\lambda \in \Lambda : |\lambda| < 1 - \tfrac{\gamma}{n}\}$$

*form a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $A^2(\mathbb{D})$.*

$$A\|p\|_2^2 \leq \sum_{\lambda \in \Lambda_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} \leq B\|p\|_2^2 \qquad \text{for all } p \in \mathcal{P}_n \,.$$

# Bergman space: from Marcinkiewicz-Zygmund families to sampling

Sequence of sets $\Lambda_n \subseteq \mathbb{D}$ converges weakly to $\Lambda \subseteq \mathbb{D}$, if for all compact disks $B \subseteq \mathbb{D}$[2]

$$\lim_{n\to\infty} d\big((\Lambda_n \cap B) \cup \partial B, (\Lambda \cap B) \cup \partial B\big) = 0\,.$$

**Theorem**

*Conversely, if $(\Lambda_n)$ is a Marcinkiewicz-Zygmund family for the polynomials $\mathcal{P}_n$ in $A^2(\mathbb{D})$, then every weak limit of $(\Lambda_n)$ is a sampling set for $A^2(\mathbb{D})$.*

[2] $d$ Hausdorff metric between compact sets in $\mathbb{D}$ w.r.t. pseudohyperbolic metric on $\mathbb{D}$

# Insights

- Prove of sampling theorems via finite-dimensional versions (=Marcinkiewicz-Zygmund families). For bandlimited functions in K.G. 1999

- Constructions of Marcinkiewicz-Zygmund families and of sampling sets are equally difficult

- Characterization of sampling sets for Bergman space by K. Seip (1995) $\Rightarrow$ Marcinkiewicz-Zygmund families in $A^2(\mathbb{D})$

## 3. Elements of the proof

# Localization of energy

For $p(z) = \sum_{k=0}^n a_k z^k \in \mathcal{P}_n$, norm on a disk $B_\rho$, $\rho < 1$, is [3]

$$\begin{aligned}
\frac{1}{\pi}\int_{B_\rho} |p(z)|^2 \, dz &= \frac{1}{\pi}\sum_{k,l=0}^{n} a_k \overline{a_l} \int_{B_\rho} z^k \bar{z}^l \, dz \\
&= \frac{1}{\pi} 2\pi \sum_{k=0}^{n} |a_k|^2 \int_0^\rho r^{2k} r dr \\
&= \sum_{k=0}^{n} |a_k|^2 \rho^{2k+2} \frac{1}{k+1} \\
&\geq \rho^{2n+2} \|p\|_{A^2}^2 .
\end{aligned}$$

Similar for $A^p(\mathbb{D}), p \neq 2$.

[3] ATTN: $\{z^k\}$ is orthogonal on all $L^2(B_\rho)$

# Localization of energy

For $p(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n$, norm on a disk $B_\rho$, $\rho < 1$, is

$$\frac{1}{\pi}\int_{B_{\rho_n}} |p(z)|^2 \, dz = \frac{1}{\pi}\sum_{k,l=0}^{n} a_k \overline{a_l} \int_{B_{\rho_n}} z^k \bar{z}^l \, dz$$

$$= \frac{1}{\pi} 2\pi \sum_{k=0}^{n} |a_k|^2 \int_0^{\rho_n} r^{2k} r dr$$

$$= \sum_{k=0}^{n} |a_k|^2 {\rho_n}^{2k+2} \frac{1}{k+1}$$

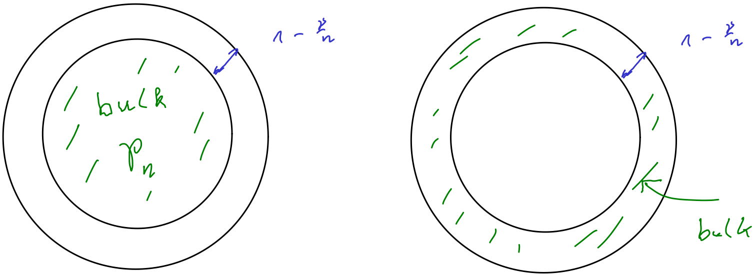
$$\geq \rho_n^{2n+2} \|p\|_{A^2}^2 \, .$$

Need

$$\rho_n^{2n+2} \geq C > 0$$

O.k. for $\rho_n = 1 - \frac{\gamma}{n}$, since $e^{-2\gamma} \leq (1 - \frac{\gamma}{n})^n \leq e^{-\gamma}$

# Localization of energy II




$\mathcal{P}_n \subseteq A^2(\mathbb{D})$ $\qquad$ $\mathcal{P}_n \subseteq H^2(\mathbb{D})$

# Comparison of kernels

Recall $k_n(z,w) = \frac{1+(n+1)(z\bar{w})^{n+2}-(n+2)(z\bar{w})^{n+1}}{(1-z\bar{w})^2}$, $k(z,w) = \frac{1}{(1-z\bar{w})^2}$

**Lemma**

*For* $\gamma > 0$

$$c_\gamma k(z,z) \le k_n(z,z) \le k(z,z) \qquad \text{for } |z| < 1 - \frac{\gamma}{n}$$

$$k_n(z,z) \asymp n^2 \qquad \text{for } |z| \ge 1 - \frac{\gamma}{n}$$

## Tail estimates

Subharmonicity of $|f(z)|^2$ yields

$$|f(\lambda)|^2 \leq \frac{1}{\pi\rho^2} \int_{B_\rho(\lambda)} |f(z)|^2 \, dz$$

Consequently for $p \in \mathcal{P}_n$ and $\Lambda$ being $\delta$-separated and $\gamma$ large enough:

$$\sum_{|\lambda| \geq 1-\gamma/n} \frac{|p(\lambda)|^2}{k(\lambda,\lambda)} \lesssim \int_{|z|\geq 1-(\gamma+\delta)/n} |p(z)|^2 \, dz \leq \epsilon \|p\|^2_{A^2}$$

# "Proof"

Assumption:

$$A\|f\|_{A^2}^2 \leq \sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{k(\lambda, \lambda)} \leq B\|f\|_{A^2}^2 \quad \text{for all } f \in A^2 .$$

For $p \in \mathcal{P}_n$

$$\sum_{\lambda \in \Lambda, |\lambda| > 1-\gamma/n} \frac{|p(\lambda)|^2}{k(\lambda, \lambda)} \lesssim \int_{|w|>1-\gamma'/n} |p(w)|^2 \, dw$$
$$< \epsilon \|p\|_{A^2}^2 .$$

$\gamma$ small enough, $\gamma' = \gamma(1+\delta)$, $\delta$ separation of $\Lambda$ w.r.t. $d$.

Consequently,

$$\begin{aligned}\sum_{\lambda\in\Lambda:|\lambda|\leq 1-\gamma/n}\frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} &\geq \sum_{\lambda\in\Lambda:|\lambda|\leq 1-\gamma/n}\frac{|p(\lambda)|^2}{k(\lambda,\lambda)}\\ &= \sum_{\lambda\in\Lambda} - \sum_{\lambda\in\Lambda:|\lambda|>1-\gamma/n} \dots \\ &\geq A\|p\|^2_{A^2} - \sum_{\lambda\in\Lambda:|\lambda|>1-\gamma/n}\frac{|p(\lambda)|^2}{k(\lambda,\lambda)} \\ &\geq (A-\epsilon)\|p\|^2_{A^2}\,.\end{aligned}$$

# 4. Fock space

## Fock space

$\mathcal{F}^2$ entire functions with norm

$$\|f\|_{\mathcal{F}^2} = \left( \int_{\mathbb{C}} |f(z)|^2 \, e^{-\pi|z|^2} dz \right)^{1/2} ,$$

- $\left\{ \left( \frac{\pi^n}{n!} \right)^{1/2} z^n : n \in \mathbb{N}_0 \right\}$ is ONB. $\Rightarrow$ Reproducing kernel for $\mathcal{P}_n$ is

$$k_n(z,w) = \sum_{k=0}^{n} \frac{(\pi z \bar{w})^k}{k!} = e^{\pi z \bar{w}} \frac{\Gamma(n+1, \pi z \bar{w})}{n!} .$$

- Reproducing kernel for $\mathcal{F}^2$ is

$$k(z,w) = \lim_{n\to\infty} k_n(z,w) = e^{\pi \bar{w} z}$$

Incomplete Gamma function $\Gamma(n+1, a) = \int_a^\infty t^n e^{-t} \, dt$ for $a \in \mathbb{C}$.

## Fock space

**Theorem**

*(i) Assume that $\Lambda \subseteq \mathbb{C}$ is a sampling set for $\mathcal{F}^2$. For $\tau > 0$ set $\rho_n$, such that $\pi \rho_n^2 = n + \sqrt{n}\tau$. Then for $\tau > 0$ large enough, $\Lambda_n = \Lambda \cap B_{\rho_n}$ form a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $\mathcal{F}^2$.*

*(ii) Assume that $(\Lambda_n)$ is a Marcinkiewicz-Zygmund family for the polynomials $\mathcal{P}_n$ in $\mathcal{F}^2$. Let $\Lambda$ be a weak limit of $(\Lambda_n)$ or some subsequence $(\Lambda_{n_k})$. Then $\Lambda$ is a sampling set for $\mathcal{F}^2$.*

- Similar proof, lots of asymptotics of incomplete Gamma function
- Correspondence Marcinkiewicz-Zygmund families $\Leftrightarrow$ sampling theorems in $\mathcal{F}^2$.

## Cardinalities of Marcinkiewicz-Zygmund families

**Corollary**

*For every $\epsilon > 0$ there exist Marcinkiewicz-Zygmund families $(\Lambda_n)$ for $\mathcal{P}_n$ in $\mathcal{F}^2$ with $\#\Lambda_n \leq (1+\epsilon)(n+1)$ points.*

Proof: Choose a sampling set $\Lambda$ for $\mathcal{F}^2$ of density $1 < D(\Lambda) < 1+\epsilon$. Then $\#(\Lambda \cap B_{\rho_n}) \leq (1+\epsilon)(n+1)$. ∎

**Proposition**

*There is no Marcinkiewicz-Zygmund family $(\Lambda_n)$ for $\mathcal{P}_n$ in $\mathcal{F}^2$ with $\#\Lambda_n = n+1$.*

Proof: Weak limit would be sampling and interpolating for $\mathcal{F}^2$ in contradiction to existing results. ∎

## Consequence for Gabor frames

Apply inverse Bargman transform $B^{-1} : \mathcal{F}^2 \to L^2(\mathbb{R})$. Then

$$B^{-1}\mathcal{P}_n = \mathrm{span}\,\{h_k : k = 0, \dots, n\}$$
$$B^{-1}(k_z) = \pi(z)h_0$$

[$h_k(t) = c_k e^{\pi t^2} \frac{d^k}{dt^k}(e^{-2\pi t^2})$ $k$-th Hermite function
$\pi(x+iy)h_0(t) = e^{-2\pi i y t} e^{-\pi(t-x)^2}$ time-frequency shift of Gaussian.]

**Theorem**

*If $\Lambda$ is sampling for $\mathcal{F}^2$ (equivalently, $\mathcal{G}(h_0, \Lambda)$ is Gabor frame in $L^2(\mathbb{R})$), then $\{\pi(\lambda)h_0 : \pi|\lambda|^2 \leq n + \sqrt{n}\tau\}$ is a frame for $V_n = \mathrm{span}\,\{h_k : k = 0, \dots, n\}$ with bounds independent of $n$.*

# 5. Hardy space

# Hardy space

Hardy space $H^2(\mathbb{D}) \subseteq L^2(\mathbb{T})$ analytic functions on $\mathbb{D}$ with norm

$$\|f\|_{H^2} = \left(\int_0^1 |f(e^{2\pi i t})|^2 \, dt\right)^{1/2} .$$

- $\{z^k : k \in \mathbb{N}_0\}$ is ONB. Reproducing kernel for $\mathcal{P}_n$ is

$$k_n(z,w) = \sum_{k=0}^{n} (z\bar{w})^n = \frac{1-(z\bar{w})^{n+1}}{1-z\bar{w}} .$$

- Reproducing kernel for $H^2$ is

$$k(z,w) = \frac{1}{1-z\bar{w}}$$

Basic fact (P. Thomas): *a function $f \in H^2(\mathbb{D})$ satisfying $A\|f\|_{H^2}^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 k(\lambda,\lambda)^{-1} \leq B\|f\|_{H^2}^2$ must be identical zero.*

$\Rightarrow$ No analogue of results for Bergman and Fock spaces

## Marcinkiewicz-Zygmund families in Hardy space

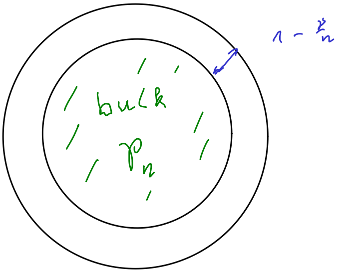
**Theorem**

*Assume that the family* $(\widetilde{\Lambda_n}) = \left(\{e^{i\nu_{n,k}} : k = 1, \ldots, L_n\}\right) \subseteq \mathbb{T}$ *is a Marcinkiewicz-Zygmund family for* $\mathcal{P}_n$ *on the torus, i.e.,*

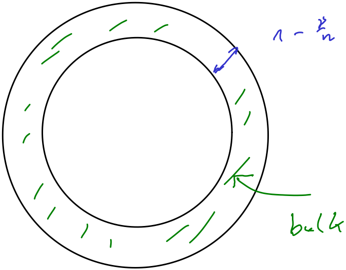
$$A\|p\|^2_{L^2(\mathbb{T})} \leq \sum_{k=1}^{L_n} \frac{|p(e^{i\nu_{n,k}})|^2}{n} \leq B\|p\|^2_{L^2(\mathbb{T})}$$

*for all polynomials p of degree n.*
*Fix* $\gamma > 0$ *arbitrary, choose* $\rho_{n,k} \in [1 - \frac{\gamma}{n}, 1)$ *arbitrary, and set* $\Lambda_n = \{\rho_{n,k} e^{i\nu_{n,k}} : k = 1, \ldots, L_n\}$ *for* $n \in \mathbb{N}$*. Then* $(\Lambda_n)$ *is a Marcinkiewicz-Zygmund family for* $\mathcal{P}_n$ *in* $H^2(\mathbb{T})$*.*

# Localization of energy III

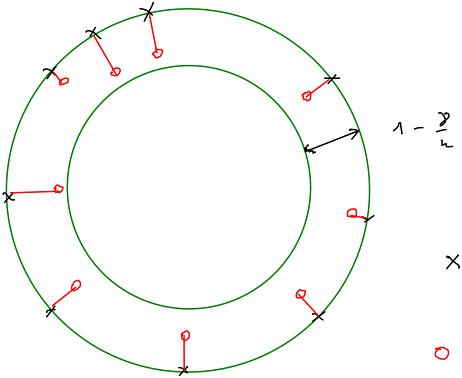



$$\mathcal{P}_n \subseteq A^2(\mathbb{D})$$




$$\mathcal{P}_n \subseteq H^2(\mathbb{D})$$

# From $\mathbb{T}$ to $\mathbb{D}$




$\mathcal{P}_n \subseteq H^2(\mathbb{D})$

$\times$ .... MZ $\subseteq L^2(\mathbb{T})$

$\circ$ MZ $\subseteq H^2(\mathbb{D})$

## For Hardy space

Marcinkiewicz-Zygmund families for $L^2(\mathbb{T})$: KG (1999), Ortega-Cerdà-Saludes (2007), etc.

Localization: Set $p_\rho(z) = p(\rho z)$. Then

$$\|p\|_{H^2}^2 \geq \|p_\rho\|_{H^2}^2 = \sum_{k=0}^{n} |a_k|^2 \rho^{2k} \geq \rho^{2n} \|p\|_{H^2}^2 \,,$$

For $\rho_n^{2n} \geq C$ choose $\rho_n = 1 - \gamma/n$.

## Outlook

- Construction of Marcinkiewicz-Zygmund families is alternative method to study sampling theorems in infinite dimensions.
- Marcinkiewicz-Zygmund families for multivariate Bergman spaces $A^2(\mathbb{B}_n)$ in $n$ complex variables on unit ball in $\mathbb{C}^n$ (attention: different notions of degree)
- Marcinkiewicz-Zygmund families for polynomials on other domains, e.g., for Bergman space on ellipse (attention: monomials no longer simultaneous orthogonal basis)
- Marcinkiewicz-Zygmund families for polynomials in Fock spaces with more general weight $e^{-Q(z)}$.
- Marcinkiewicz-Zygmund families for general reproducing kernel Hilbert spaces
- …

# References

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Thank you!