Hilbert matrix operator on Bergman-type spaces

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The Hilbert matrix

- The Hilbert matrix is an infinite matrix $\mathrm{H} = \left[\frac{1}{n+k+1}\right]_{n,k=0}^{\infty}$.

$$\mathrm{H} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \ddots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \ddots & \\ \frac{1}{4} & \frac{1}{5} & & \ddots & & \\ \frac{1}{5} & & \ddots & & & \\ & \ddots & & & & \end{bmatrix}$$

Hilbert matrix on ℓ^p spaces

- The Hilbert matrix H can be viewed as an operator on sequence spaces. Namely, if $\{a_n\}_{n=0}^{\infty}$ is a complex sequence, then:

$$\mathrm{H} : \{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right\}_{n=0}^{\infty}.$$

- Let $0 < p \leq \infty$ and define ℓ^p to be the set of complex sequences $a = \{a_n\}_{n=0}^{\infty}$ for which:

$$\|a\|_{\ell^p} = \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{\frac{1}{p}} < \infty \text{ for } 0 < p < \infty;$$

$$\|a\|_{\ell^\infty} = \sup_{n \in \mathbb{N}_0} |a_n| < \infty.$$

Theorem

Hilbert matrix H is bounded on ℓ^p if and only if $1 < p < \infty$ and

$$\|\mathrm{H}\|_{\ell^p \to \ell^p} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

Hilbert matrix on spaces of holomorphic functions

- Let $\mathrm{Hol}(\mathbb{D})$ be the space of all functions holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- The Hilbert matrix H can be viewed as an operator on spaces of holomorphic functions in the unit disc $\mathbb{D}$ by its action on their Taylor coefficients.
- If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

is a holomorphic function in the unit disc $\mathbb{D}$, then

$$\mathrm{H}f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

Hardy and mixed norm spaces

- Integral means of order p. Let $f \in \mathrm{Hol}(\mathbb{D})$, $0 < p \leq \infty$ and $0 \leq r < 1$, then:
  - $\mathrm{M}_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) \right|^p \mathrm{d}\theta \right)^{\frac{1}{p}}$ for $0 < p < \infty$;
  - $\mathrm{M}_\infty(r, f) = \sup_{0 \leq \theta < 2\pi} \left| f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) \right|$.
- The Hardy space H^p is defined as follows:

$$\mathrm{H}^p = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\mathrm{H}^p} = \sup_{0 \leq r < 1} \mathrm{M}_p(r, f) < \infty \right\}.$$

- The mixed norm space $\mathrm{H}^{p,q,\alpha}$, $0 < p, q \leq \infty$, $0 < \alpha < \infty$ is the space of all functions $f \in \mathrm{Hol}(\mathbb{D})$ for which:
  - $\|f\|_{\mathrm{H}^{p,q,\alpha}} = \left(\int_0^1 (1-r)^{q\alpha - 1} \mathrm{M}_p^q(r, f)\, \mathrm{d}r \right)^{\frac{1}{q}} < \infty$ for $0 < q < \infty$;
  - $\|f\|_{\mathrm{H}^{p,\infty,\alpha}} = \sup_{0 \leq r < 1} (1-r)^\alpha \mathrm{M}_p(r, f) < \infty$.
  - The spaces $\mathrm{H}^{p,\infty,\alpha}$ are known as weighted Hardy spaces.

Generalized mixed norm spaces

- If $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D})$ and $t\in\mathbb{R}$, we define:

$$\mathrm{D}^t f(z)=\sum_{n=0}^{\infty}(n+1)^t a_n z^n, \quad z\in\mathbb{D}.$$

- The generalized mixed norm space $\mathrm{H}_t^{p,q,\alpha}$ is the space of all functions $f\in\mathrm{Hol}(\mathbb{D})$ for which

$$\left\|\mathrm{D}^t f\right\|_{\mathrm{H}^{p,q,\alpha}}<\infty.$$

- So we can write $\mathrm{H}_t^{p,q,\alpha}=\mathrm{D}^{-t}\mathrm{H}^{p,q,\alpha}$.
- It is a well-known fact that if $f\in\mathrm{Hol}(\mathbb{D})$, $0<p,q\le\infty$, $0<\alpha,\beta<\infty$ and $t,s\in\mathbb{R}$ are such that $\alpha-t=\beta-s$, then $\left\|\mathrm{D}^t f\right\|_{\mathrm{H}^{p,q,\alpha}}$ is comparable to $\|\mathrm{D}^s f\|_{\mathrm{H}^{p,q,\beta}}$ and consequently

$$\mathrm{H}_t^{p,q,\alpha}=\mathrm{H}_s^{p,q,\beta}.$$

Weighted Bergman and Dirichlet spaces

- For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space is defined as follows:

$$\mathrm{A}^p_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\mathrm{A}^p_\alpha} = \left(\frac{\alpha+1}{\pi} \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{dm}(z) \right)^{\frac{1}{p}} < \infty \right\},$$

  where dm is the Euclidean area measure in the complex plane, that is

$$\mathrm{dm}(z) = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta, \text{ for } z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}.$$

- We note that if $\alpha = 0$ then $\mathrm{A}^p = \mathrm{A}^p_0$ are standard unweighted Bergman spaces.
- It is a well-known fact that

$$\mathrm{A}^p_\alpha = \mathrm{H}^{p,p,\frac{\alpha+1}{p}}.$$

- The weighted Dirichlet space $\mathcal{D}^p_\alpha$ is the space of all functions $f \in \mathrm{Hol}(\mathbb{D})$ for which $f' \in \mathrm{A}^p_\alpha$ or equivalently $\mathrm{D}^1 f \in \mathrm{A}^p_\alpha$. Therefore

$$\mathcal{D}^p_\alpha = \mathrm{D}^{-1}\mathrm{A}^p_\alpha = \mathrm{D}^{-1}\mathrm{H}^{p,p,\frac{\alpha+1}{p}} = \mathrm{H}_1^{p,p,\frac{\alpha+1}{p}}.$$

Hardy-Bloch and Besov spaces

- Hardy-Bloch spaces $\mathcal{B}^{p,q}$ $(0 < p, q \leq \infty)$:

$$\mathcal{B}^{p,q} = \mathrm{H}_1^{p,q,1}.$$

  - Bloch space $\mathcal{B}$: $\mathcal{B} = \mathcal{B}^{\infty,\infty} = \mathrm{H}_1^{\infty,\infty,1}$.
  - Bloch space $\mathcal{B}$ is the space of all functions $f \in \mathrm{Hol}(\mathbb{D})$ for which $\mathrm{D}^1 f \in \mathrm{H}^{\infty,\infty,1}$ or equivalently $f' \in \mathrm{H}^{\infty,\infty,1}$.
  - This implies $\mathcal{B} = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : |f'(z)| = \mathrm{O}\left(\frac{1}{1-|z|^2}\right)\right\}$.

- Besov spaces B^p $(0 < p < \infty)$:

$$\mathrm{B}^p = \mathrm{H}^{p,p,1}_{1+\frac{1}{p}}.$$

  - Note that for $1 < p < \infty$ we may write:

$$\mathrm{B}^p = \mathrm{H}^{p,p,1}_{1+\frac{1}{p}} = \mathrm{H}_1^{p,p,1-\frac{1}{p}} = \mathrm{D}^{-1}\mathrm{H}^{p,p,1-\frac{1}{p}} = \mathrm{D}^{-1}\mathrm{A}^p_{p-2},$$

which implies that in this case B^p is the set of all functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathrm{D}^1 f \in \mathrm{A}^p_{p-2}$ or equivalently $f' \in \mathrm{A}^p_{p-2}$.

Hilbert matrix on Hardy spaces

- Diamantopoulos, Siskakis (Studia Math.) (2000)
  - Hilbert matrix H is bounded on H^p if and only if $1 < p < \infty$.
  - Integral representation of the Hilbert matrix:
    - $\mathrm{H}f(z) = \int_0^1 \frac{f(t)}{1-tz}\,\mathrm{d}t$;
    - $\mathrm{H}f(z) = \int_0^1 \mathrm{T}_t f(z)\,\mathrm{d}t = \int_0^1 \frac{1}{1-(1-t)z} f\left(\frac{t}{1-(1-t)z}\right)\mathrm{d}t$.
  - If $2 \le p < \infty$, then $\|\mathrm{H}\|_{\mathrm{H}^p \to \mathrm{H}^p} \le \frac{\pi}{\sin\frac{\pi}{p}}$.
  - If $1 < p < 2$, then $\|\mathrm{H}\|_{\mathrm{H}^p \to \mathrm{H}^p} \le \frac{2\pi}{\sin\frac{\pi}{p}} + \left\|\log\frac{1}{1-z}\right\|_{\mathrm{H}^p}$.

Dostanić, Jevtić, Vukotić (J. Funct. Anal.) (2008)

Let $1 < p < \infty$. Then

$$\|\mathrm{H}\|_{\mathrm{H}^p \to \mathrm{H}^p} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

Hilbert matrix on Bergman spaces (I)

- Diamantopoulos (Illinois J. Math.) (2004)
  - Hilbert matrix H is bounded on A^p if and only if $2 < p < \infty$.
  - If $4 \leq p < \infty$, then $\|\mathrm{H}\|_{\mathrm{A}^p \to \mathrm{A}^p} \leq \dfrac{\pi}{\sin \frac{2\pi}{p}}$.
  - If $2 < p < 4$, then $\|\mathrm{H}\|_{\mathrm{A}^p \to \mathrm{A}^p} \leq \left(\dfrac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{\frac{1}{p}} \dfrac{\pi}{\sin \frac{2\pi}{p}}$.

Dostanić, Jevtić, Vukotić (J. Funct. Anal.) (2008)

Let $4 \leq p < \infty$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p \to \mathrm{A}^p} = \frac{\pi}{\sin \frac{2\pi}{p}}.$$

Conjecture

Let $2 < p < 4$. Then $\|\mathrm{H}\|_{\mathrm{A}^p \to \mathrm{A}^p} = \frac{\pi}{\sin \frac{2\pi}{p}}$.

Hilbert matrix on Bergman spaces (II)

- Božin, K. (J. Funct. Anal.) (2018)

Theorem

Let $2 < p < 4$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p \to \mathrm{A}^p} = \frac{\pi}{\sin \frac{2\pi}{p}}.$$

  - We base our method on a new way to use monotonicity of integral means and reduce the conjecture about norm to some inequalities for Beta function.
  - Inequality for Beta function

$$\left(\frac{4-p}{2} + \frac{p-2}{2} s^4\right) \mathrm{B}\left(\frac{2}{p}, 1 - \frac{2}{p}\right) \leq \int_0^1 \psi_p(t) \max\left\{s^2, t^2\right\}^{p-2} \mathrm{d}t,$$

  where $2 < p < 4$, $s \in [0,1]$ and $\psi_p(t) = t^{\frac{2}{p}-1}(1-t)^{-\frac{2}{p}}$.
    - Sturm theorem from classical theory of polynomials.
  - Hypergeometric functions.

- Lindström, Miihkinen, Wikman (Proc. Amer. Math. Soc.) (2019)
  - They provided a partially new and simplified proof of the above inequality.

Hilbert matrix on generalized mixed norm spaces (I)

- Lanucha, Nowak, Pavlović (Ann. Acad. Sci. Fenn. Math.) (2012)
  - Hilbert matrix H is bounded on $\mathrm{H}^{p,\infty,\alpha}$ if and only if $\alpha + \frac{1}{p} < 1$.
- Galanopoulos, Girela, Peláez, Siskakis (Ann. Acad. Sci. Fenn. Math.) (2014)
  - If $1 < \alpha + 2 < p$, then Hilbert matrix H is bounded on A^p_α.
  - If $p > 1$ and $p - 2 < \alpha \leq p - 1$, then Hilbert matrix H is bounded on $\mathcal{D}^p_\alpha$.

Problem

Characterize the boundedness of the Hilbert matrix H on weighted Bergman spaces A^p_α and weighted Dirichlet spaces $\mathcal{D}^p_\alpha$.

- Note that $\mathrm{A}^p_\alpha = \mathrm{H}^{p,p,\frac{\alpha+1}{p}}$ and $\mathcal{D}^p_\alpha = \mathrm{H}_1^{p,p,\frac{\alpha+1}{p}}$.

Problem (more general version)

Characterize the boundedness of the Hilbert matrix H on generalized mixed norm spaces $\mathrm{H}^{p,q,\alpha}_t$.

Hilbert matrix on generalized mixed norm spaces (II)

- Jevtić, K. (J. Math. Anal. Appl.) (2017)

Theorem

Hilbert matrix H is bounded on generalized mixed norm space $\mathrm{H}_t^{p,q,\alpha}$ if and only if

$$0 < \frac{1}{p} + \alpha - t < 1.$$

- Consequences:
  - H is bounded on A_α^p if and only if $1 < \alpha + 2 < p$.
  - H is bounded on $\mathcal{D}_\alpha^p$ if and only if $\max\{-1, p-2\} < \alpha < 2p - 2$.
  - H is bounded on $\mathcal{B}^{p,q} = \mathrm{H}_1^{p,q,1}$ if and only if $1 < p < \infty$.
  - H is unbounded on Bloch space $\mathcal{B} = \mathcal{B}^{\infty,\infty} = \mathrm{H}_1^{\infty,\infty,1}$.
  - H is unbounded on Besov space $\mathrm{B}^p = \mathrm{H}_{1+\frac{1}{p}}^{p,p,1}$.

Hilbert matrix on weighted Bergman spaces (I)

- Hilbert matrix H is bounded on weighted Bergman space A^p_α if and only if $1 < \alpha + 2 < p$.
- K. (Glasgow Math. J.) (2018)

Theorem

Let $1 < \alpha + 2 < p$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} \geq \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}.$$

Proof (sketch). Let $1 < \gamma < \alpha + 2 < p$ and $f_\gamma(z) = (1-z)^{-\frac{\gamma}{p}}$, where $z \in \mathbb{D}$.

- $\|f_\gamma\|^p_{\mathrm{A}^p_\alpha} = {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma}{2}, \alpha+2; 1\right)$.
- $\mathrm{H}f_\gamma(z) = \int_0^1 \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma}{p}}(1-tz)} = \mathrm{B}\left(1, 1-\frac{\gamma}{p}\right) {}_2F_1\left(1, 1, 2-\frac{\gamma}{p}; z\right)$.

We note that here B denotes Beta function and ${}_2F_1$ denotes classical Gauss hypergeometric function.

Hilbert matrix on weighted Bergman spaces (II)

- Proof (sketch) (continued). Based on the previous formulas, the following can be concluded:
  - $\|f_\gamma\|_{\mathrm{A}^p_\alpha} < \infty$;
  - $\lim\limits_{\gamma\to\alpha+2} \|f_\gamma\|_{\mathrm{A}^p_\alpha} = \infty$;
  - $\mathrm{H} f_\gamma(z) = \dfrac{\pi}{\sin\frac{\pi\gamma}{p}} \left(f_\gamma(z) + g_\gamma(z)\right)$, where $\sup\limits_{1<\gamma<\alpha+2} \|g_\gamma\|_{\mathrm{A}^p_\alpha} \leq \mathrm{C}_{p,\alpha} < \infty$.

  Then

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha\to\mathrm{A}^p_\alpha} \geq \frac{\|\mathrm{H} f_\gamma\|_{\mathrm{A}^p_\alpha}}{\|f_\gamma\|_{\mathrm{A}^p_\alpha}} \geq \frac{\pi}{\sin\frac{\pi\gamma}{p}} \frac{\|f_\gamma\|_{\mathrm{A}^p_\alpha} - \|g_\gamma\|_{\mathrm{A}^p_\alpha}}{\|f_\gamma\|_{\mathrm{A}^p_\alpha}} = \frac{\pi}{\sin\frac{\pi\gamma}{p}} \left(1 - \frac{\|g_\gamma\|_{\mathrm{A}^p_\alpha}}{\|f_\gamma\|_{\mathrm{A}^p_\alpha}}\right).$$

  Letting $\gamma \to \alpha + 2$, we obtain

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha\to\mathrm{A}^p_\alpha} \geq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

  This completes the proof. ■

Hilbert matrix on weighted Bergman spaces (III)

- K. (Glasgow Math. J.) (2018)

Conjecture

Let $1 < \alpha + 2 < p$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} = \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}.$$

  - If $\alpha > 0$ and $2(\alpha+2) \leq p$, then $\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} \leq \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}$.The conjecture is solved in this case.
  - As we have noticed, the conjecture is solved in the case $\alpha = 0$, that is, in the case of unweighted Bergman spaces A^p.
  - Unsolved cases: $\alpha > 0,\ \alpha + 2 < p < 2(\alpha + 2)$ and $-1 < \alpha < 0,\ \alpha + 2 < p$.
- Lindström, Miihkinen, Wikman (Ann. Acad. Sci. Fenn. Math.) (2020)
  - They solved the conjecture in the case when

$$\alpha > 0 \text{ and } \alpha + 2 + \sqrt{(\alpha+2)^2 - \frac{1}{2}(\alpha+2)} \leq p < 2(\alpha+2).$$

Hilbert matrix on weighted Bergman spaces (IV)

- K. (J. Geom. Anal.) (2021)
  - $\alpha > 0,\ \alpha + 2 < p < 2(\alpha+2)$. Let

$$\Phi_\alpha(x) = 2x^2 - (4(\alpha+2)+1)\,x + 2\sqrt{\alpha+2}\sqrt{x} + \alpha + 2,$$

where $x \in (\alpha+2, 2(\alpha+2))$. It is easy to check that Φ_α is an increasing function on interval $(\alpha+2, 2(\alpha+2))$. Also

$$\Phi_\alpha(\alpha+2) < 0 \text{ and } \Phi_\alpha(2(\alpha+2)) > 0.$$

This means that function Φ_α has a unique zero α_0 on the interval $(\alpha+2, 2(\alpha+2))$. Moreover, we get $\Phi_\alpha < 0$ on $(\alpha+2, \alpha_0)$ and $\Phi_\alpha > 0$ on $(\alpha_0, 2(\alpha+2))$.

Theorem

Let $\alpha > 0$ and $\alpha_0 \le p < 2(\alpha+2)$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Hilbert matrix on weighted Bergman spaces (V)

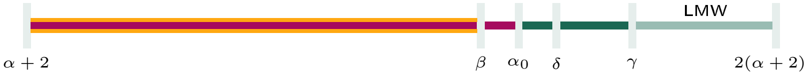
- Location of α_0. Let $\alpha > 0$ and let

$$\beta = \alpha + 2 + \sqrt{(\alpha+2)^2 - (\alpha+2)};$$

$$\gamma = \alpha + 2 + \sqrt{(\alpha+2)^2 - 0.5\,(\alpha+2)};$$

$$\delta = \alpha+2+\sqrt{(\alpha+2)^2 - \left(\sqrt{2} - \frac{1}{2}\right)(\alpha+2)} \approx \alpha+2+\sqrt{(\alpha+2)^2 - 0.914\,(\alpha+2)}.$$

  - It is easy to check that $\Phi_\alpha(\beta) < 0$ and $\Phi_\alpha(\delta) > 0$.



Hilbert matrix on weighted Bergman spaces (VI)

- K. (J. Geom. Anal.) (2021)
  - On the interval $(\alpha+2, \beta]$ we have the following partial result.

Theorem

Let $\alpha > 0$, $\alpha + 2 < p \leq \beta$ and suppose that the following inequality holds:

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t \leq \frac{1}{\alpha+1}\mathrm{B}\left(\frac{\alpha+2}{p}, 1-\frac{\alpha+2}{p}\right), \qquad (*)$$

where

$$\psi_{p,\alpha}(t) = t^{\frac{\alpha+2}{p}-1}(1-t)^{-\frac{\alpha+2}{p}} \text{ and } \xi_{p,\alpha}(t) = \int_{\left(\frac{t}{2-t}\right)^2}^1 \rho^{\frac{p}{2}-(\alpha+2)}(1-\rho)^\alpha\,\mathrm{d}\rho,$$

for $0 < t < 1$. Then

$$\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Hilbert matrix on weighted Bergman spaces (VII)

- K. (J. Geom. Anal.) (2021)
  - We note that inequality $(*)$ is not always satisfied under the given conditions. Namely, a calculation involving *Mathematica* shows that when $\alpha = 1$ ($\beta \approx 5.449$, $\alpha_0 \approx 5.487$) and $p = 4.4$ we have

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t - \frac{1}{\alpha+1}\mathrm{B}\left(\frac{\alpha+2}{p}, 1 - \frac{\alpha+2}{p}\right) \approx 0.962 > 0.$$

    On the other hand, if $\alpha = 1$ and $p = 5.2$ then

$$\int_0^1 \psi_{p,\alpha}(t)\xi_{p,\alpha}(t)\,\mathrm{d}t - \frac{1}{\alpha+1}\mathrm{B}\left(\frac{\alpha+2}{p}, 1 - \frac{\alpha+2}{p}\right) \approx -0.103 < 0.$$

  - In the remaining unsolved case $-1 < \alpha < 0,\ \alpha + 2 < p$ we only have the following result.

Theorem

Let $-1 < \alpha < 0$ and $p > \alpha + 2$.

(i) If $p \geq 2(\alpha+2)$ then $\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} \leq 2^{\frac{\alpha+2}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}$.

(ii) If $\alpha + 2 < p < 2(\alpha+2)$ then $\|\mathrm{H}\|_{\mathrm{A}^p_\alpha \to \mathrm{A}^p_\alpha} \leq 2^{\frac{\alpha+2}{p}} \left(1 + 2^{\frac{2(\alpha+2)}{p} - 1}\right) \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}$.

Generalized Hilbert matrix

- We note $\mathrm{H} = \left[\frac{1}{n+k+1}\right]_{n,k=0}^{\infty} = \left[\int_0^1 t^{n+k}\, \mathrm{d}t\right]_{n,k=0}^{\infty}$.
- Let μ be a positive Borel measure on $[0,1)$. The generalized Hilbert matrix is an infinite matrix

$$\mathrm{H}_\mu = \left[\int_0^1 t^{n+k}\, \mathrm{d}\mu(t)\right]_{n,k=0}^{\infty}.$$

  - Chatzifountas, Girela, Peláez (J. Math. Anal. Appl.) (2014) - *A generalized Hilbert matrix acting on Hardy spaces.*
  - Galanopoulos, Peláez (Studia Math.) (2014) - *A Hankel matrix acting on Hardy and Bergman spaces.*
  - Girela, Merchán (Integral Equations Operator Theory) (2017) - *A Hankel matrix acting on spaces of analytic functions.*
  - Girela, Merchán (Banach J. Math. Anal.) (2018) - *A generalized Hilbert operator acting on conformally invariant spaces.*
  - Girela, Merchán (Rev. Mat. Comput.) (2019) - *Hankel matrices acting on the Hardy space H^1 and on Dirichlet spaces.*
  - Merchán (Collect. Math.) (2019) - *Lipschitz spaces and a generalized Hilbert operator.*
  - Jevtić, K. (Complex Anal. Oper. Theory) (2019) - *Generalized Hilbert matrices acting on spaces that are close to the Hardy space H^1 and to the space $BMOA$.*

• THANK YOU •