Composition of analytic paraproducts

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- Benyi-Maldonado-Naibo (2010): A paraproduct is a bilinear, noncommutative form Λ that satisfies product reconstruction $fg = \Lambda(f,g) + \Lambda(g,f)$, (up to smooth errors), a Hölder-type inequality, and a Leibniz-type rule.

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Primary Aim

Study the boundedness of compositions of analytic paraproducts on A^p_α

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$$\leq \|f\|^{\frac{1}{2}}_{\alpha,p} \|g' T_g f\|^{\frac{1}{2}}_{\alpha+p,p} \simeq \|T_g^2 f\|^{\frac{1}{2}}_{\alpha,p}, \quad \|f\|_{\alpha,p} = 1.$$

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Lemma (Aleman-Cima-Constantin)

Let $0 < p < \infty$, $\alpha \geq -1$, $r, q, s > 0$, $\frac{1}{r} + \frac{1}{s} = \frac{1}{q}$, and $g \in A^r_\alpha$. Then $T_g : A^s_\alpha \to A^q_\alpha$ is bounded and there exists a constant $c > 0$, independent of g, satisfying that $\|T_g\|_{A^s_\alpha \to A^q_\alpha} \leq c\|g\|_{\alpha,r}$.

Proposition

Let $g \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$, $\alpha \geq -1$ and let P be a polynomial of degree $n \geq 1$. If $P(T_g)$ is bounded on A^p_α, then T_g is bounded on A^p_α.

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$$|a_n| c_n \| T_g \|^n_{\alpha,p} \leq |a_n| \| T_g^n \|_{\alpha,p} \leq c_{n,p} \sum_{k=0}^{n-1} |a_k| \| T_g \|^k_{\alpha,p} + \| P(T_g) \|_{\alpha,p}.$$

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so they are bounded on A^p_α if and only if $g^2 \in \mathscr{B}$, $\alpha > -1$ ($g^2 \in BMOA$, $\alpha = -1$).

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In particular, if $g^n \in BMOA$ ($g^n \in \mathcal{B}$), then $g^m \in BMOA$ ($g^m \in \mathcal{B}$).

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$g^2 \in \mathscr{B}(BMOA) \Rightarrow M_g Tg$ and $S_g T_g$ are bounded on A^p_α

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Vector space structure of $\mathscr{A}_g$. The ST-decomposition

Any g-operator $L \neq 0$ can be written as

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where $n \in \mathbb{N} \cup \{0\}$ and $P_0, \dots, P_{n+1}$ are unique polynomials such that either $P_n \neq 0$ or $P_{n+1} \neq 0$.

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- The existence of $\{P_k\}_{k=0}^{n+1}$ is proved by an induction process based on the formula

$$T_g S_g = S_g T_g - T_g^2$$

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where $n \in \mathbb{N} \cup \{0\}$ and $P_0, \dots, P_{n+1}$ are unique polynomials such that either $P_n \neq 0$ or $P_{n+1} \neq 0$.

- The existence of $\{P_k\}_{k=0}^{n+1}$ is proved by an induction process based on the formula

$$T_g S_g = S_g T_g - T_g^2$$

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Let $g \in \mathscr{H}(\mathbb{D})$, $0 < p < \infty$ and $\alpha \geq -1$. Let $L_g \neq 0$ be a g-operator. If L_g is bounded on A^p_α, then T_g is bounded on A^p_α.

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\begin{aligned}
& c_N |a_N| \|T_{g_r}\|^N_{\alpha,p} - c'_{N,p} \sum_{k=0}^{N-1} |a_k| \|T_{g_r}\|^k_{\alpha,p} \\
& \leq \|Q_n(T_{g_r})\|_{\alpha,p} \\
& = \|[L_{g_r}, T_{g_r}]_n\|_{\alpha,p} \leq c_{n,p} \|L_{g_r}\|_{\alpha,p} \|T_{g_r}\|^n_{\alpha,p}.
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A byproduct: Composition of two paraproducts

Boundedness of composition of analytic paraproducts on A^p_α, $\alpha > -1$			
	T_g	S_g	M_g
T_g	$T_g^2 \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g \in \mathscr{B}$	$S_g T_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g^2 \in \mathscr{B}$	$M_g T_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g^2 \in \mathscr{B}$
S_g	$T_g S_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g^2 \in \mathscr{B}$	$S_g^2 \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g \in H^\infty$	$M_g S_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g \in H^\infty$
M_g	$T_g M_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g^2 \in \mathscr{B}$	$S_g M_g \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g \in H^\infty$	$M_g^2 \in \mathrm{B}(A^p_\alpha) \Leftrightarrow g \in H^\infty$

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- The previous result shows that the boundedness of g-operators with ST-representation such that $P_{n+1}=0$, and $P_n(0)=0$ cannot be characterized with conditions of the form $g\in H^\infty$, or $g^n\in\mathcal{B}(BMOA)$, with $n\in\mathbb{N}$.