Weighted Hardy spaces and their composition operators

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Introduction

This work is concerned with the boundedness of composition operators $C_\varphi,\ f \mapsto f \circ \varphi$, with $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $\mathbb{D} = \{|z| < 1\}$. The set of such φ (symbols) is denoted S. We denote $S_0 = \{\varphi \in S : \varphi(0) = 0\}$. If $g = f \circ \varphi,\ \varphi \in S_0$, g is said "subordinate to f". We have the non abelian semi-group property

$$C_{\varphi\circ\psi} = C_\psi C_\varphi \text{ for all } \varphi, \psi \in S.$$

Here, C_φ acts on weighted Hilbert spaces of analytic functions

$$H = H^2(\beta) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n; \sum_{n=0}^{\infty} |a_n|^2 \beta_n =: \|f\|_H^2 < \infty\}$$

$$\text{where } \beta = (\beta_n)_{n\geq 0} \text{ with } \beta_n > 0,\ \liminf_n \beta_n^{1/n} \geq 1.$$

$H^2(\beta) \hookrightarrow \mathcal{H}(\mathbb{D})$; it has ONB $(z^n/\sqrt{\beta_n})_{n\geq 0}$ and a reproducing kernel :

$$f(a) = \langle f,\ K_a\rangle,\ a \in \mathbb{D},\ K_a(z) = \sum_{n=0}^{\infty} \frac{(\bar{a}z)^n}{\beta_n} \in H.$$

Examples

Set $dA(z) = \frac{dxdy}{\pi}$, let $\alpha > -1$. The norm in the Dirichlet space $\mathcal{D}_\alpha$ is

$$\|f\|^2_{\mathcal{D}_\alpha} = |f(0)|^2 + (\alpha+1)\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha dA(z)$$

$$=: |f(0)|^2 + D_\alpha(f).$$

$$\beta_n = (\alpha+1)\int_0^1 n^2 t^{n-1}(1-t)^\alpha dt \approx (n+1)^{1-\alpha}.$$

The norm in the Bergman space $B_\alpha = \mathcal{D}_{\alpha+2}$ is

$$\|f\|^2_{B_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dA(z)$$

$$\beta_n = (\alpha+1)\int_0^1 t^n(1-t)^\alpha dt \approx (n+1)^{-1-\alpha}$$

Examples

Set $dA(z) = \frac{dxdy}{\pi}$, let $\alpha > -1$. The norm in the Dirichlet space $\mathcal{D}_\alpha$ is

$$\|f\|_{\mathcal{D}_\alpha}^2 = |f(0)|^2 + (\alpha+1)\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dA(z)$$

$$=: |f(0)|^2 + D_\alpha(f).$$

$$\beta_n = (\alpha+1)\int_0^1 n^2 t^{n-1}(1-t)^\alpha dt \approx (n+1)^{1-\alpha}.$$

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Observe that, for Bergman spaces, (β_n) is completely monotone (cf. Hausdorff moment theorem), which is rather restrictive.
The seminal result is Littlewood's subordination principle.

Littlewood's subordination principle

Take $\beta_n \equiv 1$, $i.e.$ $H = H^2$. It holds :

Theorem (Littlewood, 1925)

Let $\varphi \in S_0$. Then

$$C_\varphi : H^2 \to H^2 \text{ contractively.}$$

Equivalently, the matrix $(a_{m,n}) = (\widehat{\varphi^n}(m))$ has norm ≤ 1 on $\ell^2 = \ell^2(\mathbb{N}_0)$.
Indeed, $C_\varphi(z^n) = \varphi^n(z) = \sum_{m=0}^{\infty} \widehat{\varphi^n}(m) z^m$.

Theorem (Folklore ?)

Let $\varphi_a \in S$ be the involutive automorphism $z \mapsto \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$. Then

$$C_{\varphi_a} : H^2 \to H^2 \text{ with } \|C_{\varphi_a}\| = \sqrt{\frac{1+|a|}{1-|a|}}.$$

Any $\varphi \in S$ writes $\varphi = \varphi_a \circ \psi$ with $\psi \in S_0$ and $C_\varphi = C_\psi C_{\varphi_a}$, so $C_\varphi : H^2 \to H^2$ for all those φ.

Boundedness on H^2, pursued

The proofs are "different". For Littlewood, with the backward shift B, $Bf(z) = \frac{f(z)-f(0)}{z} = \sum_{k=0}^{\infty} c_{k+1} z^k$ and $M_\varphi(g) = \varphi g,\ \|M_\varphi\|_{H^2 \to H^2} \leq 1$:

$$C_\varphi f = f(0) + M_\varphi C_\varphi B f, \text{ hence } \|C_\varphi f\|^2 \leq |f(0)|^2 + \|C_\varphi B f\|^2.$$

(reminds Maurey-Pisier's extrapolation principle).
Iterate this, with f a polynomial of degree $\leq n$, so that $B^{n+1} f = 0$:

$$\|C_\varphi f\|^2 \leq \sum_{j=0}^{n} |B^j f(0)|^2 + \|C_\varphi B^{n+1} f\|^2 = \sum_{j=0}^{n} |B^j f(0)|^2 = \|f\|^2.$$

For "Folklore", take $0 < a < 1$. Then a Toeplitz matrix shows up :

$$C_\varphi^* C_\varphi = (a^{|m-n|}) = (\widehat{P_a}(m-n)), \text{ so that } \|C_\varphi\|^2 = \|P_a\|_\infty.$$

P_a is the Poisson kernel at a. Works for φ inner with $\varphi(0) = a$, via

$$\langle \varphi^m, \varphi^n \rangle = \langle \varphi^{m-n}, 1 \rangle = a^{m-n},\ m \geq n.$$

Both proofs use integral representations.

Boundedness on H^2, the end

Theorem (Rogosinski)

Let $\varphi \in S_0$, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = (f \circ \varphi)(z) = \sum_{k=0}^{\infty} b_k z^k$. Then

$$\sum_{k=0}^{n} |b_k|^2 \leq \sum_{k=0}^{n} |a_k|^2, n = 0, 1, \ldots$$

Set in general $S_n(\sum_{k=0}^{\infty} c_k z^k) = \sum_{k=0}^{n} c_k z^k$. Just observe that, thanks to $\varphi(0) = 0$, $S_n(g) = S_n C_\varphi S_n(f)$, hence $\|S_n(g)\|^2 \leq \|S_n(f)\|^2$.

Theorem (Goluzin)

Let φ be a symbol in S_0. Assume that (β_n) is non-increasing. Then

$$C_\varphi : H^2(\beta) \to H^2(\beta) \text{ contractively.}$$

Rogosinski plus Abel summation.

What if $\varphi(0) \neq 0$? if (β_n) does not essentially decrease ?

Boundedness on $H^2(\beta)$

1. All C_{φ_a} can be bounded, with even $D_0(f \circ \varphi_a) \equiv D_0(f)$, but not all C_φ (Dirichlet space $\mathcal{D}$, $\beta_n = n+1$).
Arazy-Fisher-Peetre (1985) studied in depth the semi-normed spaces (X, ρ) for which $\rho(f \circ \varphi_a) \equiv \rho(f) \quad \forall \varphi_a$, isometrically.

2. If all C_φ are bounded, $\sup_n \beta_n < \infty$, because then $H^\infty \hookrightarrow H^2(\beta) = H$ via $\varphi = C_\varphi(z)$ and so $\beta_n = \|z^n\|_H^2 \leq C\|z^n\|_\infty^2 = C$.

3. When $\beta \downarrow$, boundedness of $C_\varphi, \varphi \in S$, boils down to that of the C_{φ_a}.

4. But even so, some C_{φ_a} can be unbounded on H, e.g. $\beta_n = \exp(-\sqrt{n})$ (Kriete-Mac Cluer). More generally, we could prove :

Theorem (LLQR 0)

Fix $0 < a < 1$, let $T_a(z) = -\varphi_{-a}(z) = \frac{z+a}{1+az}$. Then :
1. $C_{T_a} : H \to H$ bounded implies $\|K_z\|_H \ll (1-|z|)^{-\alpha'}$.
2. Equivalently, $C_{T_a} : H \to H$ bounded implies $\beta_n \geq \delta n^{-\alpha}, n \geq 1$.

This necessary condition of polynomial decay for β_n is not sufficient.

Boundedness on $H^2(\beta)$, pursued

We present here such a NSC for boundedness of all C_φ when $\beta = (\beta_n)$ is essentially decreasing, meaning : $\tilde{\beta}_n := \sup_{m \geq n} \beta_m \leq C\beta_n$. Then $H^2(\beta) = H^2(\tilde{\beta})$, and we will mostly assume here β non increasing.
We first need a definition :
the (non increasing) sequence (β_n) has Δ_2 if (think of Orlicz spaces)

$$\beta_{2n} \geq \delta \beta_n \text{ for all } n \geq 1.$$

That condition writes as well as a slow oscillation condition :

$$1/\rho(m/n) \leq \frac{\beta_m}{\beta_n} \leq \rho(m/n)$$

where $\rho : (0, \infty) \to (0, \infty)$ is locally bounded from above and below.
Strictly stronger than polynomial decay.

Statements

We can now state four theorems. In the first two, β is non increasing.

Theorem (LLQR 1)

The Δ_2-condition is sufficient for the boundedness of all $C_\varphi : H \to H$.

Theorem (LLQR 2)

The Δ_2-condition is necessary for the boundedness of all $C_\varphi : H \to H$.

Theorem (LLQR 3)

Take a general sequence (β_n). Then TFAE :
1. β is essentially decreasing.
2.

$$\sup_{\varphi \in S_0} \|C_\varphi\| < \infty.$$

A remark

Our Theorems 1, 2, 3 answer the question if one wishes.

Corollary

Given an admissible weight β, $\liminf_{n\to\infty} \beta_n^{1/n} \geq 1$, *TFAE :*

1. β *is essentially decreasing and has* Δ_2.
2. *All* $C_\varphi : H^2(\beta) \to H^2(\beta)$ *are bounded, with moreover* $\sup_{\varphi\in S_0} \|C_\varphi\| < \infty$.

Theorem 4 to come indicates that the situation might be much more intricate.

Statements, pursued

No "Baire trick" is available in Theorem 3 ! Indeed

Theorem (LLQR4)

Assume that β has polynomial decay and that, for every $\delta > 0$, there exists $C_\delta > 0$ with

$$\beta_m \leq C_\delta \beta_n \text{ whenever } m > (1+\delta)n.$$

Then $\|C_\varphi\| < \infty$ for all $\varphi \in S_0$.
This can happen with $\sup_{\varphi \in S_0} \|C_\varphi\| = \infty$ and non essentially decreasing β.

About the assumptions : β can oscillate, but not with two indices far apart.
Each $\varphi \in S_0$ generates its own δ and C_δ.

Rest of the talk

We will give sketchy proofs of some theorems, insisting on two aspects :

- The use of the stationary phase and van der Corput method.
- The use of a result of Katsnelson (already exploited by Chalendar-Partington, more recently by LLQR under a reinforced form involving approximation numbers).
- We will discuss in a second part a result on conditional multipliers using Katsnelson's result.

Difficulty : no integral representation for β_n.

Reminder 2

Theorem (LLQR 2)

The Δ_2-condition is necessary for the boundedness of all $C_\varphi : H \to H$.

We begin with some facts on the stationary phase method.

Stationary phase

We use a version of the van der Corput inequalities, or rather the stationary phase method.

Theorem (Stationary phase)

Let $F : [A, B] \to \mathbb{R}$, C^3, with $F'' > 0$. Let c be the unique point in $]A, B[$ where $F'(c) = 0$. Assume that, with $\lambda_2, \Lambda_3, \eta > 0$, one has :

1) $[c - \eta, c + \eta] \subseteq [A, B]$;
2) $F''(x) \geq \lambda_2$ *for all* $x \in [c - \eta, c + \eta]$;
3) $|F'''(x)| \leq \Lambda_3$ *for all* $x \in [A, B]$.

Then, with absolute O :

$$I := \int_A^B \exp[i\, F(x)]\, dx = \sqrt{2\pi}\, \frac{e^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\Lambda_3\right),$$

We control the modulus and argument of I.

Specialization

With $T_a(z) = (z+a)(1+az)^{-1}$, $T_a^n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m$, $0 < a < 1$:

$$T_a(e^{ix}) = \exp[ih_a(x)] \text{ where } h_a(x) = \int_0^x P_{-a}(t)dt.$$

P_{-a} is the Poisson kernel at $-a$. Hence, since h_a is odd on $[-\pi, \pi]$:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp[i(nh_a(x) - mx)]dx = 2\Re I_{m,n}$$

where

$$I_{m,n} = \int_0^{\pi} \exp[i(nh_a(x) - mx)]dx =: \int_0^{\pi} \exp[i\,F_m(x)]dx.$$

The preceding with $\lambda_2 \approx n$, $\Lambda_3 \approx n$, $\eta = n^{-2/5}$ gives for $m \approx n$:
(note that $nP_{-a}(x_m) = m$, and $F_m''(x) = nh_a''(x)$)

$$I_{m,n} = \sqrt{2\pi}\frac{e^{i\theta_m}}{\sqrt{|F_m''(x_m)|}} + O(n^{-3/5}),\ \theta_m = F_m(x_m) + \pi/4.$$

$$\theta_{m+1} - \theta_m = -x_{m+1} + O(n^{-1}) \text{ and } |F_m''(x_m)| \approx n.$$

Proof 2

Since $-\pi/2 \leq -x_{m+1} \leq -\pi/4$, we use the preceding to get (with $\varphi = T_a = -\varphi_{-a}$ and $a_{m,n} = \widehat{\varphi^n}(m)$) :

Lemma

There exists $0 < \delta < 1/3$ satisfying : for every integer $n \geq 1$, there exists $J_n \subset \mathbb{N}$ such that

- $J_n \subset [(1-3\delta)n, (1-\delta)n]$.
- $|J_n| \geq \delta n$.
- $m \in J_n \Longrightarrow |a_{m,n}| = |\widehat{\varphi^n}(m)| = (1/\pi)|\Re I_{m,n}| \geq \delta n^{-1/2}$.

The behavior of $a_{m,n}$ is very irregular. Indeed for fixed n :

- $\sum_{m=0}^{\infty} |a_{m,n}|^2 = 1$ (Parseval for the inner function φ^n).
- $\sup_m |a_{m,n}| \ll n^{-1/3}$.

The experts on this are O. Szehr and R. Zarouf [6], see also S. Borichev, K. Fouchet, R. Zarouf with a complete and sophisticated picture, using notably the Airy function.
Our self-contained lemma above is sufficient here.
Recently, in passing, Y. Meyer reproved and used the second item.

Proof 2, the end

Now, our boundedness assumption implies, since we have $\varphi^n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m$:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|\varphi^n\|_H^2 = \|C_\varphi(z^n)\|_H^2 \leq C^2 \|z^n\|_H^2 = C^2 \beta_n.$$

This gives a general necessary condition :

$$C^2 \beta_n \geq \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \geq \sum_{m \in J_n} |a_{m,n}|^2 \beta_m \geq \delta^3 \frac{1}{|J_n|} \sum_{m \in J_n} \beta_m. \tag{1}$$

When β decreases, we get $C^2 \beta_n \geq \delta^3 \beta_{(1-\delta)n}$, and (β_n) satisfies Δ_2.

Proof 2, a variant

Instead of automorphisms, we could use the singular inner functions

$$I_a(z) = \exp\big(-a\frac{1+z}{1-z}\big) =: \sum_{m=0}^{\infty} c_m(a) z^m,\ a > 0.$$

Think of $(I_1)^n = I_n$ and of $a = n$.
Thanks once more to stationary phase, we can prove

Lemma (Szegö precised)

Let $a \geq 1$. It holds $c_m(a) = e^{-a} L_m^{(-1)}(2a)$ and

$$c_m(a) = c\, a^{1/4} m^{-3/4} \cos(d\sqrt{am} + \pi/4) + R_m(a) =: M_m + R_m(a) \tag{2}$$

where $c = \pi^{-1/2} 2^{1/4}$, $d = 2\sqrt{2} < 2\pi$, and $|R_m(a)| \leq K\sqrt{a}\, m^{-5/4}$, with K some numerical constant.

A weaker form was already used by H. Shapiro and D. Newman.
For the proof of Theorem 4, we need a transition.

A result of V. Katsnelson

Theorem (V. È. Katsnelson, 1975)

Let H be a separable Hilbert space, $(e_n)_{n\geq 0}$ an ONB of H, and $T\colon H\to H$, bounded. Set $(a_{m,n})=(\langle Te_n, e_m\rangle)$.
Let $(d_n)_{n\geq 0}$ be an arbitrary sequence of positive numbers and D the (mostly unbounded) diagonal operator with $D(e_n)=d_ne_n$, $n\geq 0$. Assume that

$$d_m < d_n \Longrightarrow a_{m,n}=0.$$

Then $D^{-1}TD\colon H\to H$ is bounded as well and moreover

$$\|D^{-1}TD\| \leq \|T\|\,. \tag{3}$$

Proved with (d_n) increasing and $(a_{m,n})$ lower triangular. The proof here is the same.
Assumption met, with $d_n=1/\sqrt{\beta_n}$, by $T=C_\varphi : H^2\to H^2$ and $D^{-1}TD=C_\varphi : H^2(\beta)\to H^2(\beta)$ as soon as $\varphi(0)=0$. Said on ℓ^2 :
If $T=(a_{m,n})$, then $D^{-1}TD=(a_{m,n}\frac{d_n}{d_m})$ and $D^{-z}TD^z=(a_{m,n}(\frac{d_n}{d_m})^z)$.

V. Katsnelson, continued

Maximum principle for the subharmonic and bounded function

$$u(z) = \|D^{-z}TD^{z}\| \text{ in } \{\Re z > 0\},\ u(iy) = \|T\|,\ u(1) = \|D^{-1}TD\|.$$

Baby example (with $a > 0,\ 0 < d_1 < d_2$) :

$$T_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$D^{-1}T_aD = T_{ad_1/d_2} \text{ and } \|T_a\| \uparrow \text{ with } a.$$

Indeed

$$\|T_a\|^2 = \frac{a^2 + 2 + a\sqrt{a^2+4}}{2}.$$

Reminder 4

Theorem (LLQR4)

Assume that β *has polynomial decay and that, for every* $\delta > 0$*, there exists* $C = C_\delta > 0$ *with*

$$\beta_m \leq C\beta_n \text{ whenever } m > (1+\delta)n.$$

Then $\|C_\varphi\| < \infty$ *for all* $\varphi \in S_0$.
This can happen with $\sup_{\varphi \in S_0} \|C_\varphi\| = \infty$ *and non essentially decreasing* β.

Let $\varphi \in S_0$, and $\delta = \delta(\varphi) > 0$, $C = C_\delta > 0$ to come. Set

$$\gamma_n = \max\big(\beta_n, \sup_{m > (1+\delta)n} \beta_m\big).$$

Our assumption implies

$$\beta_n \leq \gamma_n \leq C_\delta \beta_n \quad \text{and } m > (1+\delta)n \Longrightarrow \gamma_m \leq \gamma_n. \tag{4}$$

So that $H^2(\beta) = H^2(\gamma)$. The implied matrices on ℓ^2 are, for $C_\varphi : H^2 \to H^2$ and $C_\varphi : H^2(\gamma) \to H^2(\gamma)$ respectively :

$$A = (a_{m,n}) \text{ with } a_{m,n} = \widehat{\varphi^n}(m),$$

$$B = (b_{m,n}) := (a_{m,n}\sqrt{\gamma_m/\gamma_n}).$$

Proof 4

We first need the (Cauchy inequalities with $r = e^{-1/2}$, Schwarz lemma)

Lemma

Let $\varphi \in S_0$, with $\varphi \neq uI$, $|u| = 1$. Then for some $\rho = \rho(\varphi) > 0$

$$|a_{m,n}| := |\widehat{\varphi^n}(m)| \leq \exp\Big[-\frac{1}{2}\big((1+\rho)n - m\big)\Big].$$

The point is : $1 + \rho > 1$! We put $\delta = \rho/2$ and

$$A = (a_{m,n}) = A_1 + A_2,\ A_1 = (a_{m,n}1_{m\leq(1+\delta)n}),\ A_2 = (a_{m,n}1_{m>(1+\delta)n}).$$

A_2 is super lower-triangular.
We write accordingly

$$B = (b_{m,n}) = B_1 + B_2,\ B_1 = (b_{m,n}1_{m\leq(1+\delta)n}),\ B_2 = (b_{m,n}1_{m>(1+\delta)n}).$$

Proof 4, continued

We must show that $B = (b_{m,n}) := (a_{m,n}\sqrt{\gamma_m/\gamma_n})$ is bounded on ℓ^2, knowing that $A = (a_{m,n})$ is. Here, $B = D^{-1}AD$, $D =$ diag (d_n), $d_n = \frac{1}{\sqrt{\gamma_n}}$. We prove :

- $A_1 =: (a^{(1)}_{m,n})$ is Hilbert-Schmidt, i.e. $\sum_{m,n} |a^{(1)}_{m,n}|^2 < \infty$, and B_1 as well, since the lemma gives $|a_{m,n}| \leq e^{-(\delta/2)n}$ for $m \leq (1+\delta)n$. Moreover $\gamma_m/\gamma_n \lesssim 1/\gamma_n \lesssim n^\alpha$.
- $A_2 = A - A_1$ is bounded on ℓ^2 (obvious).
- $B_2 = D^{-1}A_2 D$ is bounded on ℓ^2 (key point !)
- $B = B_1 + B_2$ bounded on ℓ^2, as desired.

For key point, recall (4) : $m > (1+\delta)n \Longrightarrow \gamma_m \leq \gamma_n$. Now, we are allowed to use Katsnelson for $A_2 = (a_{m,n}1_{m>(1+\delta)n}) =: (a^{(2)}_{m,n})$, since

$$d_m < d_n \Longrightarrow \gamma_m > \gamma_n \Longrightarrow m \leq (1+\delta)n \Longrightarrow a^{(2)}_{m,n} = 0.$$

Proof 4, the end

As for the counterexample we define, on blocks $(k!, (k+1)!]$:

$$\beta_n = \begin{cases} 1/k! & \text{if } k! < n \leq (k+1)! - 2, \\ 1/(k+1)! & \text{if } n = (k+1)! - 1, \\ 1/k! & \text{if } n = (k+1)!. \end{cases} \tag{5}$$

The sequence (β_n) is not essentially decreasing since $\beta_{n+1}/\beta_n = k+1$ for $n = (k+1)! - 1$. It has polynomial decay since $\beta_n \geq 1/(n+1)$.
Next, $\beta_m \leq \beta_n$ as soon as $m \geq n+2$, and the assumptions of Theorem 4 are met.
Finally, $\sup_{\varphi \in S_0} \|C_\varphi\| = \infty$ by Theorem LLQR3.

Remark. On that example, no C_{φ_a} is bounded ! Because condition (2)

$$C^2 \beta_n \geq \delta^2 \frac{1}{|J_n|} \sum_{m \in J_n} \beta_m$$

fails for $n = (k+1)! - 1$.
We switch to something else.

2. Conditional multipliers

Here is one more application of Katsnelson's theorem.
The set of multipliers of the analytic Hilbert space H is :

$$\mathcal{M}(H) = \{w : wf \in H \text{ for all } f \in H\} \subset H.$$

Mostly, $\mathcal{M}(H) = H^\infty$ (not for the Dirichlet space $\mathcal{D}_0$!)

To study weighted composition operators $T = M_w C_\varphi$, $T(f) = w(f \circ \varphi)$, we need a conditional version of $\mathcal{M}(H)$ given $\varphi \in S$, namely :

$$\mathcal{M}(H, \varphi) = \{w : w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

When $C_\varphi : H \to H$, we have the double inclusion

$$(H^\infty =)\mathcal{M}(H) \subset \mathcal{M}(H, \varphi) \subset H.$$

Indeed, $w = M_w C_\varphi(1)$.
We study the extreme cases in those inclusions.

2. Conditional multipliers, statement

First a definition : the space $H = H^2(\beta)$, $\beta \downarrow$ is "admissible" (e.g. Bergman space B_α) if

$$H^2 \hookrightarrow H \text{ and } C_\varphi : H \to H \text{ for } \varphi \in \text{ Aut } \mathbb{D}.$$

This forbids e.g. $\beta_n = \exp(-\sqrt{n})$. Katsnelson implies $\mathcal{M}(H) = H^\infty$ when H is admissible.
We can now state (due to Attele, Contreras, H. D. for $H = H^2$) :

Theorem (LLQR)

Let H be admissible, and $\varphi : \mathbb{D} \to \mathbb{D}$. Then :

1. $\mathcal{M}(H^2, \varphi) \subset \mathcal{M}(H, \varphi)$.
2. $\mathcal{M}(H, \varphi) = H \iff \|\varphi\|_\infty < 1$.
3. $\mathcal{M}(H, \varphi) = H^\infty \iff \varphi$ *is a finite Blaschke product.*

We start with $\Longrightarrow$ of the third statement for $H = H^2$.
m is the Haar measure of $\mathbb{T} = \partial\mathbb{D}$. $\int$ stands for $\int_\mathbb{T}$.

2. Conditional multipliers on H^2

We have the following, denoting $\|w\|_{op}$ the norm of the operator $M_w C_\varphi$ on H^2.

Theorem (Attele)

It holds $\|K_a\|_{op} \leq \sqrt{\frac{1+|\varphi(a)|}{1-|\varphi(a)|}} \frac{1}{\sqrt{1-|a|^2}}$.
Hence, if $\mathcal{M}(H^2, \varphi) = H^\infty$, *we have*

$$\|K_a\|_\infty \leq C\|K_a\|_{op} \quad \textit{and finally } \frac{1-|\varphi(a)|}{1-|a|} \leq C' = 2C^2.$$

Proof. Let $b = \varphi(a)$. Use $(1-|a|^2)|K_a|^2 = |\varphi_a'|$ and $\varphi_b \circ \varphi \circ \varphi_a \in S_0$:

$$(1-|a|^2) \int |K_a|^2 |f \circ \varphi|^2 dm = \int |\varphi_a'| \, |f \circ \varphi \circ \varphi_a \circ \varphi_a|^2 dm$$

$$= \int |f \circ \varphi \circ \varphi_a|^2 dm = \int |f \circ \varphi_b \circ \varphi_b \circ \varphi \circ \varphi_a|^2 dm$$

$$\leq \int |f \circ \varphi_b|^2 dm \leq \frac{1+|b|}{1-|b|} \int |f|^2 dm.$$

2. Conditional multipliers on H^2, next

For the last statement, the inequality $\|K_a\|_\infty \leq C\|K_a\|_{op}$ reads

$$\frac{1}{1-|a|} \leq C\sqrt{\frac{2}{1-|\varphi(a)|}}\frac{1}{\sqrt{1-|a|}}.$$

Then simplify. But this last statement characterizes the set $\mathcal{F}$ of finite Blaschke products ! More precisely :

$$\varphi \in \mathcal{F} \Longrightarrow \frac{1-|\varphi(a)|}{1-|a|} \leq C.$$

$$\lim_{|a|\to 1} |\varphi(a)| = 1 \Longrightarrow \varphi \in \mathcal{F}.$$

(note, see Contreras and H. D., that $\mathcal{M}(H^2, \varphi) \subset H^p$ for some $p > 2$ would suffice). Or even $\mathcal{M}(H, \varphi) \subset H^\psi$ with $\lim_{x\to\infty} x^{-2}\psi(x) = \infty$.

We no longer have precise estimates on $\|K_a\|_{op}$ for admissible spaces $H = H^2(\beta)$, but we will have the inclusion 1. of Theorem LLQR.

2.Conditional multipliers, the end

This inclusion $\mathcal{M}(H^2,\varphi) \subset \mathcal{M}(H,\varphi)$ provides :

$$\mathcal{M}(H,\varphi) = H^\infty \Longrightarrow \varphi \text{ is a finite Blaschke product.}$$

Here is the proof of the above inclusion. Assume first that $\varphi(0) = 0$.
Let $w \in \mathcal{M}(H^2,\varphi)$. Katsnelson for the operator $T = M_w C_\varphi$ is licit !

$$\langle T(e_n), e_m\rangle_{H^2} = \langle w\varphi^n, e_m\rangle_{H^2} = 0 \text{ for } m < n.$$

This gives $w \in \mathcal{M}(H,\varphi)$, and $\mathcal{M}(H^2,\varphi) \subset \mathcal{M}(H,\varphi)$ in that case.
In the general case, set $a = \varphi(0)$, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Our admissibility assumption implies

$$\mathcal{M}(H^2,\varphi) = \mathcal{M}(H^2,\varphi_a\circ\varphi) \subset \mathcal{M}(H,\varphi_a\circ\varphi) = \mathcal{M}(H,\varphi),$$

since $(\varphi_a\circ\varphi)(0) = 0$ and $C_{\varphi_a} : H \to H$ and $H^2 \to H^2$.

The rest is then fairly easy.

Some questions

We finally mention some "open questions".

1. Is there a complex-variable free proof of Katsnelson's result ?
2. Can the result be more dependent on the specific value of (d_n) ?
3. A theorem of Grothendieck (Grothendieck's constant) describes Schur multipliers $\lambda_{m,n}$ of bounded matrices. Namely

$$(a_{m,n}) \text{ bounded } \Longrightarrow (a_{m,n}\lambda_{m,n}) \text{ bounded} .$$

   We must have $\lambda_{m,n} = \langle x_m, y_n \rangle$ where $\sup \|x_m\| < \infty$, $\sup \|y_n\| < \infty$. Katsnelson says that $\lambda_{m,n} = \frac{d_n}{d_m}$ is a Schur multiplier of lower-triangular bounded matrices. Can we hope for a full characterization of Schur multipliers for those matrices ?
4. Is the slow oscillation of the bounded sequence (β_n) a NSC in the general case ?

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