The Berezin Transform on the Bergman Space

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Workshop on Bergman Spaces

Focus Program on Analytic Function Spaces and their Applications

Fields Institute

$\mathbb{D}$ denotes open unit disk in $\mathbf{C}$.
dA denotes area measure on $\mathbb{D}$, normalized so that $dA(\mathbb{D}) = 1$.

Definition: *Bergman space*

The *Bergman space* L^2_a is the set of analytic functions h on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |h|^2 \, dA < \infty.$$

With the inner product

$$\langle h, g \rangle = \int_{\mathbb{D}} h \, \overline{g} \, dA,$$

the Bergman space L^2_a is a Hilbert space.

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the Bergman space L^2_a is a Hilbert space.

For each $z \in \mathbb{D}$, the point evaluation map

$$h \mapsto h(z)$$

is a bounded linear functional on L^2_a.
Thus for each $z \in \mathbb{D}$, $\exists k_z \in L^2_a$ such that

$$h(z) = \langle h, k_z \rangle$$

for all $h \in L^2_a$. (z = zee = zed)
k_z is called the *reproducing kernel* at z for L^2_a.
If $z, w \in \mathbb{D}$, then

$$k_z(w) = \frac{1}{(1 - \overline{z} w)^2}.$$

Thus if $h \in L^2_a$ and $z \in \mathbb{D}$, then

$$h(z) = \int_{\mathbb{D}} \frac{h(w)}{(1 - z\overline{w})^2} \, dA(w).$$

Berezin transform

For $z \in \mathbb{D}$, the reproducing kernel k_z satisfies

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For $z \in \mathbb{D}$ the *normalized reproducing kernel* $K_z \in L_a^2$ and is defined by

$$K_z = \frac{k_z}{\|k_z\|_2}.$$

Thus $\|K_z\|_2 = 1$ for all $z \in \mathbb{D}$.

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Definition: *Berezin transform* [5]

For a bounded linear operator

$$S\colon L_a^2 \to L_a^2,$$

the *Berezin transform* of S is the function

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Note that

$$|\widetilde{S}(z)| = |\langle SK_z, K_z \rangle| \le \|SK_z\|_2 \, \|K_z\|_2 \le \|S\|$$

for all $z \in \mathbb{D}$.

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If S is a bounded operator on L_a^2, then

- $|\widetilde{S}(z)| \leq \|S\|$ for all $z \in \mathbb{D}$;
- $\widetilde{S}$ is a real analytic function on $\mathbb{D} \subset \mathbf{R}^2$;
- $\widetilde{S^*} = \overline{\widetilde{S}}$.
- The map

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Thus $\widetilde{S}$ contains all information about S.

What properties of S can be deduced from properties of $\widetilde{S}$?

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Claim: $K_z \to 0$ weakly in L^2_a as $|z| \uparrow 1$.

In other words, for every $h \in L^2_a$, we have $\langle h, K_z \rangle \to 0$ as $|z| \uparrow 1$.

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Proof: Suppose $h \in L^2_a$ and $\epsilon > 0$. There exists a bounded analytic function f on $\mathbb{D}$ such that $\|h - f\|_2 < \epsilon$. Now

$$\begin{aligned}
|\langle h, K_z \rangle| &\le |\langle h - f, K_z \rangle| + |\langle f, K_z \rangle| \\
&\le \|h - f\|_2 + (1 - |z|^2)|f(z)| \\
&< \epsilon + (1 - |z|^2)\|f\|_\infty \\
&< 2\epsilon
\end{aligned}$$

for $|z|$ sufficiently close to 1. ∎

Compact operators send sequences converging weakly to 0 to sequences converging to 0 in norm.

If $S\colon L^2_a \to L^2_a$ is compact, then

$$\widetilde{S}(z) \to 0 \text{ as } |z| \uparrow 1.$$

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Example: Define $S\colon L_a^2 \to L_a^2$ by

$$(Sf)(z) = f(-z).$$

Then S is not compact. In fact, S is unitary.

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Example: Define $S\colon L_a^2 \to L_a^2$ by

$$(Sf)(z) = f(-z).$$

Then S is not compact. In fact, S is unitary. However,

$$\begin{aligned}\widetilde{S}(z) &= (1-|z|^2)^2\langle Sk_z, k_z\rangle \\ &= (1-|z|^2)^2 (Sk_z)(z) \\ &= \frac{(1-|z|^2)^2}{(1+|z|^2)^2} \quad \text{because} \quad k_z(w) = \frac{1}{(1-\overline{z}w)^2} \\ &\to 0 \text{ as } |z| \uparrow 1.\end{aligned}$$

Toeplitz operators

Let $P\colon L^2(\mathbb{D}) \to L^2_a$ be the orthogonal projection of $L^2(\mathbb{D})$ onto L^2_a.
If $u \in L^2(\mathbb{D})$ and $z \in \mathbb{D}$, then

$$\begin{aligned}(Pu)(z) &= \langle Pu, k_z \rangle \\ &= \langle u, k_z \rangle \\ &= \int_{\mathbb{D}} \frac{u(w)}{(1 - z\overline{w})^2}\, dA(w).\end{aligned}$$

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For $f \in L^\infty(\mathbb{D})$, define the *Toeplitz operator* $T_f\colon L^2_a \to L^2_a$ by

$$T_f h = P(fh).$$

Clearly

$$\|T_f\| \leq \|f\|_\infty.$$

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$$\begin{aligned}\widetilde{T_f}(z) &= \langle T_f K_z, K_z\rangle \\ &= \langle P(fK_z), K_z\rangle \\ &= \langle fK_z, K_z\rangle \\ &= (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \overline{z}w|^4}\, dA(w).\end{aligned}$$

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Definition: *Berezin transform of function*

For $f \in L^\infty(\mathbb{D})$, the *Berezin transform* of f is the function $\widetilde{f}\colon \mathbb{D} \to \mathbf{C}$ defined by

$$\widetilde{f} = \widetilde{T_f}.$$

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$$1 = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{1}{|1 - \overline{z}w|^4} \, dA(w)$$

for all $z \in \mathbb{D}$, think of $\widetilde{f}(z)$ as a weighted average of the values of $f(w)$, with most of the weight near z if $|z| \approx 1$.

Thus if $f \in C(\overline{\mathbb{D}})$, then $\widetilde{f} \in C(\overline{\mathbb{D}})$ and

$$\widetilde{f}|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}.$$

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Coburn [6]

Suppose $f \in C(\overline{\mathbb{D}})$. Then the following are equivalent:

1. T_f is compact.
2. $f|_{\partial\mathbb{D}} = 0$.
3. $\widetilde{f}(z) \to 0$ as $|z| \uparrow 1$.

Berezin transforms of harmonic functions

$$\widetilde{f}(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \overline{z}w|^4} \, dA(w)$$

Let H^∞ = bounded analytic functions on $\mathbb{D}$. If $f \in H^\infty$ and $h \in L^2_a$, then

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Thus the real part of each function in H^∞ equals its Berezin transform. This leads to:

If u is a bounded harmonic function on $\mathbb{D}$, then $\widetilde{u} = u$.

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Engliš [8]

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Ahern, Flores, Rudin [1]: The Englis result above holds on the unit ball of $\mathbf{C}^n$ if and only if $n \leq 11$.

back to Toeplitz operators

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$$(T_f h)(z) = \int_{\mathbb{D}} \frac{f(w)h(w)}{(1 - z\overline{w})^2}\, dA(w)$$

for $f \in L^\infty(\mathbb{D})$, $h \in L^2_a$, and $z \in \mathbb{D}$.

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If S is a finite sum of finite products of Toeplitz operators then the following are equivalent:

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Hankel operators

Recall that P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto L^2_a. Thus $I - P$ is the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2(\mathbb{D}) \ominus L^2_a$, which is the orthogonal complement of L^2_a.

For $g \in L^\infty(\mathbb{D})$, the *Hankel operator*

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$$\begin{aligned}(H_g h)(z) &= g(z)h(z) - \big(P(gh)\big)(z)\\ &= \int_{\mathbb{D}} \frac{\big(g(z) - g(w)\big)h(w)}{(1 - z\overline{w})^2}\, dA(w).\end{aligned}$$

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$$= \int_{\mathbb{D}} \frac{\big(g(z) - g(w)\big)h(w)}{(1 - z\overline{w})^2}\, dA(w).$$

For which $g \in L^\infty(\mathbb{D})$ is H_g compact? Note that H_g does not map L^2_a into L^2_a and thus it does not make sense to take the Berezin transform of H_g. However,

$$H_g \text{ is compact} \iff {H_g}^* H_g \colon L^2_a \to L^2_a \text{ is compact.}$$

If $f, g \in L^\infty(\mathbb{D})$, then

$$H_{\overline{f}}^* H_g = T_{\overline{f}g} - T_{\overline{f}} T_g.$$

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Suppose $f \in H^\infty$. Then

$$H_{\overline{f}} \text{ is compact } \iff \lim_{|z|\uparrow 1} (1-|z|^2) f'(z) = 0.$$

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There exist Blaschke products f with infinitely many zeros such that $\lim_{|z|\uparrow 1}(1-|z|^2)f'(z) = 0$ (see [12]). Thus there exist Blaschke products f with infinitely many zeros such that $H_{\overline{f}}$ is compact.

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Proof in one direction:

$H_{\overline{f}}$ is compact

$\iff$ ${H_{\overline{f}}}^* H_{\overline{f}}$ is compact

$\iff$ $\widetilde{{H_{\overline{f}}}^* H_{\overline{f}}}(z) \to 0$ as $|z| \uparrow 1$

$\iff$ $\langle {H_{\overline{f}}}^* H_{\overline{f}} K_z, K_z \rangle \to 0$ as $|z| \uparrow 1$

$\iff$ $\|H_{\overline{f}} K_z\|_2 \to 0$ as $|z| \uparrow 1$

$\iff$ $\|\overline{f} K_z - P(\overline{f} K_z)\|_2 \to 0$ as $|z| \uparrow 1$

$\iff$ $\|\overline{f} K_z - \overline{f(z)} K_z\|_2 \to 0$ as $|z| \uparrow 1$

$\iff$ $\|(f - f(z)) K_z\|_2 \to 0$ as $|z| \uparrow 1$.

bound on $(1-|z|^2)f'(z)$

Suppose $g = \sum_{n=0}^{\infty} a_n z^n \in L_a^2$. Then

$$\|g - g(0)\|_2^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n+1} \geq \frac{|a_1|^2}{2} = \frac{|g'(0)|^2}{2}.$$

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If $z \in \mathbb{D}$ and $f \in H^\infty$ then

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Use $(*)$ with $g = f \circ \varphi_z$, getting

$$(1-|z|^2)|f'(z)| \leq \sqrt{2}\|f \circ \varphi_z - f(z)\|_2$$
$$= \sqrt{2}\|(f - f(z))K_z\|_2.$$

Thus if $f \in H^\infty$ and $H_{\overline{f}}$ is compact, then

$$\lim_{|z|\uparrow 1}(1-|z|^2)f'(z) = 0.$$

Bloch space and little Bloch space

Definition: *Bloch space*

The *Bloch space* $\mathcal{B}$ is the set of analytic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The *little Bloch space* $\mathcal{B}_0$ is the set of analytic functions f on $\mathbb{D}$ such that

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The Bloch space $\mathcal{B}$ is a Banach space and $\mathcal{B}_0$ is a closed subspace with the norm

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For $g \in L^2(\mathbb{D})$, define the Hankel operator H_g from H^∞ (with the L^2_a-norm) to L^2_a by

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Suppose $f \in L^2_a$. Then

- $H_{\overline{f}}$ is bounded if and only if $f \in \mathcal{B}$.
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- The Möbius invariant space generated by H^2 is BMOA.
- The Möbius invariant space generated by L^2_a is $\mathcal{B}$.

Coifman, Rochberg, and Weiss [7]

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Furthermore, if $\varphi \colon L_a^1 \to \mathbf{C}$ is a bounded linear functional, then $\exists$ a unique $f \in \mathcal{B}$ such that

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$$P\big(L^\infty(\mathbb{D})\big) = \mathcal{B}.$$

Also,

$$P\big(C(\overline{\mathbb{D}})\big) = \mathcal{B}_0.$$

Littlewood on Bloch's theorem

Littlewood ([10], page 145):

> "Bloch's theorem. One of the queerest things in mathematics, and one might judge that only a madman could do it. He was aiming at an elementary proof of Picard's theorem, an impudently damn fool idea. With this as a start it is a just reasonable stroke of insight to conjecture Bloch's theorem. The result once conjectured (and being true), a proof was, of course, bound to emerge sooner or later. But, to keep up the air of farce to the end, the proof itself is crazy."

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THANK YOU!