

Dirichlet spaces, inner functions and isometric composition operators

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Part I

Möbius invariant function spaces
generated by Dirichlet spaces

Classical spaces

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

A=Area measure

Definition (Dirichlet space \mathcal{D})

$$f \in \mathcal{D} \iff \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

Definition (Hardy space H^2)

$$f \in H^2 \iff \|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) < +\infty.$$

Definition (BMOA)

$$f \in BMOA \iff \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{|1 - \bar{w}z|}{|z - w|} dA(z) < +\infty.$$



Dirichlet spaces with superharmonic weights (A. Aleman, 1993)

- $\omega : \mathbb{D} \mapsto (0, +\infty]$, positive superharmonic function

$$\begin{aligned}\omega(z) &= \int_{\mathbb{D}} \log \left| \frac{1 - \bar{w}z}{z - w} \right| d\mu(w) + \int_{\partial\mathbb{D}} \frac{1 - |\zeta|^2}{|\zeta - z|^2} d\nu(\zeta) \\ &= U_\mu(z) + P_\nu(z),\end{aligned}$$

$$\int_{\mathbb{D}} (1 - |z|) d\mu(z) < +\infty, \quad \text{and} \quad \nu(\partial\mathbb{D}) < +\infty.$$

Definition (Weighted Dirichlet space \mathcal{D}_ω)

$$f \in \mathcal{D}_\omega \iff \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < +\infty.$$

- Dirichlet spaces \mathcal{D}_ν with harmonic weights, $\omega = P_\nu$,
(S. Richter, 1991)

We will concentrate on

- Dirichlet spaces \mathcal{D}_μ ,

$$\omega(z) = U_\mu(z) = \int_{\mathbb{D}} \log \left| \frac{1 - \bar{w}z}{z - w} \right| d\mu(w).$$

- $\lim_{r \rightarrow 1} U_\mu(r\zeta) = 0$ for almost every $\zeta \in \partial\mathbb{D}$.

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$$\|f\|_{\mathcal{D}_\mu}^2 = \|f\|_{H^2}^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z).$$

Examples (\mathcal{D}_p spaces with radial superharmonic weights)

- $\omega_p(z) = (1 - |z|^2)^p$, $p \in (0, 1)$,
- $d\mu_p = -\Delta((1 - |z|^2)^p) dA(z)$,
- $\mu_p(\mathbb{D}) = +\infty$.

Definition (Carleson measures)

For every arc $I \subset \partial\mathbb{D}$ with length $|I|$,

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1, \zeta \in I\}.$$

μ is Carleson measure (for H^2) if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty.$$

Theorem (JAMS, with G. Bao, J. Mashreghi and H. Wulan)

- $\mathcal{D}_\mu \subset H^2$, $\forall \mu$,
- if $\mu(\mathbb{D}) < +\infty$, $BMOA \subset \mathcal{D}_\mu \subset H^2$,
- if $(1 - |z|^2)d\mu(z)$ is a Carleson measure, $\mathcal{D} \subsetneq \mathcal{D}_\mu$.

Definition (The Möbius invariant function space $M(\mathcal{D}_\mu)$)

The Möbius invariant function space $M(\mathcal{D}_\mu)$ generated by \mathcal{D}_μ is the class of holomorphic functions f on \mathbb{D} , with

$$\|f\|_{M(\mathcal{D}_\mu)} = \sup_{\phi \in \text{Aut}(\mathbb{D})} \|f \circ \phi - f(\phi(0))\|_{\mathcal{D}_\mu} < \infty.$$

Examples

- $M(H^2) = BMOA$,
 - $M(\mathcal{D}) = \mathcal{D}$,
 - $M(\mathcal{D}_p) = \mathcal{Q}_p$, $p \in (0, 1)$.
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- J. Xiao, *Geometric \mathcal{Q}_p Functions*, Birkhäuser Verlag, 2006.
 - H. Wulan and K. Zhu, *Möbius invariant \mathcal{Q}_K spaces*, Springer, 2017.

Theorem (JAMS, with G. Bao, J. Mashreghi and H. Wulan)

- If $\mu(\mathbb{D}) < +\infty$, $M(\mathcal{D}_\mu) = BMOA$.
- If $\mu(\mathbb{D}) = +\infty$, the following are equivalent:
 - (1) $M(\mathcal{D}_\mu)$ is not trivial,
 - (2) $\mathcal{D} \subset M(\mathcal{D}_\mu)$,
 - (3) $(1 - |z|^2)d\mu(z)$ is a Carleson measure.

Which inner functions are contained in $M(\mathcal{D}_\mu)$ ($\mu(\mathbb{D}) = +\infty$)?

Definition

$\phi \in H^2$ is called inner if $|\phi(\zeta)| = 1$ for almost every $\zeta \in \partial\mathbb{D}$.

Definition (Carleson-Newman Blaschke products)

A Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k}z}$$

is called Carleson-Newman Blaschke product if $\sum_{k=1}^{\infty} (1 - |a_k|^2) \delta_{a_k}$ is a Carleson measure.

Theorem (JAMS, with G. Bao, J. Mashreghi and H. Wulan)

Suppose that $\mu(\mathbb{D}) = +\infty$ and let I be an inner function.

- ① If $I \in M(\mathcal{D}_\mu)$, I is a Blaschke product.
- ② Suppose that I is a Carleson-Newman Blaschke product with zeros $\{a_k\}_{k=1}^\infty$. Then $I \in M(\mathcal{D}_\mu)$ if and only if

$$\sup_{\phi \in \text{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} \left(1 - \left| \frac{a_k - \phi(w)}{1 - \overline{a_k} \phi(w)} \right|^2 \right) d\mu(w) < \infty.$$

Proof. Let $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$.

$\nu = t\delta_1$, $t > 0$,

$$S_\nu(z) = \exp\left(-t \frac{1+z}{1-z}\right)$$

$$|S_\nu(z)| = \exp\left(-t \frac{1-|z|^2}{|1-z|^2}\right)$$

$$S_\nu \notin M(\mathcal{D}_\mu)$$

Fix $c > 0$. Consider the horodisk

$$D_c = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|1 - z|^2} > c \right\},$$

note that

$$|S_\nu| \leq e^{-tc}, \quad \text{on } D_c,$$

and let

$$\mu_a = \mu \circ \sigma_a, \quad a \in \mathbb{D}.$$

$$\begin{aligned}
\int_{\mathbb{D}} |(S_\nu \circ \sigma_a)'(z)|^2 U_\mu(z) dA(z) &= \int_{\mathbb{D}} (1 - |S_\nu(\sigma_a(z))|^2) d\mu(z) \\
&\geq \int_{\sigma_a(D_c)} (1 - |S_\nu(\sigma_a(z))|^2) d\mu(z) \\
&= \int_{D_c} (1 - |S_\nu(z)|^2) d\mu_a(z) \\
&\geq (1 - e^{-2tc}) \mu(\sigma_a(D_c)).
\end{aligned}$$

Let $\phi_r(z) = -\sigma_r(z)$ and note that $\phi_r(D_c) \nearrow \mathbb{D}$ as $r \rightarrow 1$. Then

$$\lim_{r \rightarrow 1} \|S_\nu \circ \phi_r\|_{\mathcal{D}_\mu}^2 \geq \lim_{r \rightarrow 1} (1 - e^{-2tc}) \mu(\phi_r(D_c)) = (1 - e^{-2tc}) \mu(\mathbb{D}) = +\infty.$$

$$S_\nu \notin M(\mathcal{D}_\mu).$$

Part II

Isometric composition operators on BMOA

Let $D \subset \mathbb{C}$ be a domain and let $y \in D$.

Definition

The Green function $G_D(\cdot, y) : D \mapsto (0, +\infty]$, with pole at $y \in D$,

- is harmonic on $D \setminus \{y\}$,
- $x \mapsto G_D(x, y) - \log \frac{1}{|x-y|}$, is harmonic on D ,
-

$$\lim_{x \rightarrow \zeta} G_D(x, y) = 0,$$

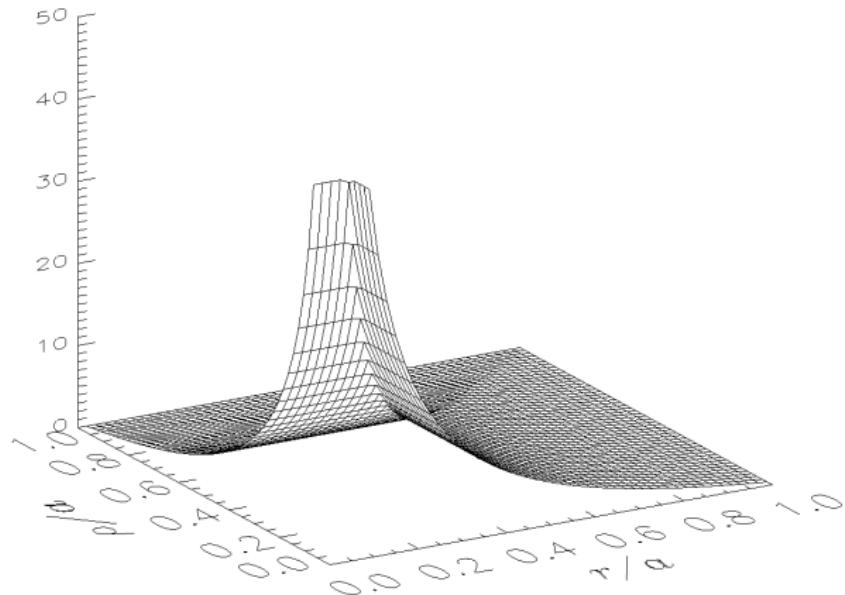
for every $\zeta \in \partial D$ except on a subset of zero logarithmic capacity

Examples

$$\mathbb{D} = \{x \in \mathbb{C} : |x| < 1\},$$

$$G_{\mathbb{D}}(x, y) = \log \left| \frac{1 - x\bar{y}}{x - y} \right|, \quad x, y \in \mathbb{D}.$$

Green function of a rectangle with pole at 0.



Theorem (Strict monotonicity)

Suppose that D and Ω are Greenian domains such that $D \subset \Omega$ and $\Omega \setminus D$ has positive logarithmic capacity. Then

$$G_D(z, w) < G_\Omega(z, w).$$

Theorem (Subordination Principle for Green function)

Let $f : D \rightarrow \Omega$ be a holomorphic function and suppose that D and Ω are Greenian domains. Then

$$G_D(z, w) \leq G_\Omega(f(z), f(w))$$

Equality holds for two distinct points $z_0, w_0 \in D$ if and only if f is injective and $\Omega \setminus f(D)$ has zero logarithmic capacity.

Theorem (Lindelöf Principle)

Let $f : D \rightarrow \Omega$ be a holomorphic function and suppose that D and Ω are Greenian domains. Then

$$G_{\Omega}(u_0, f(x)) \geq \sum_{f(a)=u_0} G_D(a, x),$$

where $x \in D$ and $u_0 \in \Omega$.

Essentially, equality holds only for universal covering maps and inner functions (Betsakos, CMFT 2014).

When $D = \Omega = \mathbb{D}$ and $\phi : \mathbb{D} \rightarrow \mathbb{D}$, we obtain the

Theorem (Littlewood inequality)

$$N_\phi(w, z) \leq \log \left| \frac{1 - \bar{w}\phi(z)}{w - \phi(z)} \right|, \quad z, w \in \mathbb{D},$$

where

$$N_\phi(w, z) := \sum_{\phi(a)=w} \log \left| \frac{1 - \bar{a}z}{a - z} \right|, \quad w \in \mathbb{D},$$

is the Nevanlinna counting function of ϕ with respect to $z \in \mathbb{D}$.
Equality holds if and only if ϕ is an inner function.

Definition

For $p > 0$, the Hardy space H^p contains the holomorphic functions on \mathbb{D} satisfying

$$\|f\|_p := \left(\sup_{r \in (0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty.$$

Definition

$BMOA = M(H^p)$, $p \geq 1$, and

$$\|f\|_{BMOA_p} := |f(0)| + \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_p < +\infty,$$

where

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D},$$

are equivalent norms on $BMOA$.

Definition

Let $\phi : \mathbb{D} \mapsto \mathbb{D}$ be a holomorphic function. The composition operator C_ϕ is defined by

$$C_\phi(f) = f \circ \phi,$$

for every holomorphic function f on \mathbb{D} .

Isometric composition operators:

- Hardy spaces: F. Forelli, M. J. Martín and D. Vukotić.
- Dirichlet space: M. J. Martín and D. Vukotić.
- Bergman spaces: M. J. Martín and D. Vukotić.
- Bloch space: F. Colonna, M. J. Martín and D. Vukotić.
- Bloch type spaces: N. Zorboska.
- Besov type spaces: M. A. Shabazz and M. Tjani.
- General Banach spaces of analytic functions: A. Mas and D. Vukotić.
- BMOA: J. Laitila.

- J. Laitila, *Isometric composition operators on BMOA*, Math. Nachr. 283, No. 11, 1646–1653 (2010).

Theorem (J. Laitila, 2010)

Suppose that $\phi(0) = 0$. The following are equivalent:

- $\|C_\phi(f)\|_{BMOA_2} = \|f\|_{BMOA_2}$, for every $f \in BMOA$.
- For every $w \in \mathbb{D}$, there exists a sequence $(a_n) \subset \mathbb{D}$ such that $\phi(a_n) \rightarrow w$ and

$$\|\sigma_{\phi(a_n)} \circ \phi \circ \sigma_{a_n}\|_2 \rightarrow 1.$$

Problem (J. Laitila, 2010)

Let $p \geq 1$. Characterize all holomorphic functions $\phi : \mathbb{D} \mapsto \mathbb{D}$ for which

$$\|C_\phi(f)\|_{BMOA_p} = \|f\|_{BMOA_p}, \text{ for every } f \in BMOA.$$

Theorem (BLMS (2021), S.P.)

Suppose that $\phi(0) = 0$. For $p \in [1, 2]$, the following are equivalent:

- $\|C_\phi(f)\|_{BMOA_p} = \|f\|_{BMOA_p}$, for every $f \in BMOA$.
- For every $w \in \mathbb{D}$, there exists a sequence $(a_n) \subset \mathbb{D}$ such that $\phi(a_n) \rightarrow w$ and

$$\lim_{n \rightarrow \infty} \left(\log \left| \frac{1 - \bar{z}\phi(a_n)}{z - \phi(a_n)} \right| - N_\phi(z, a_n) \right) = 0,$$

for almost every $z \in \mathbb{D}$.

Theorem (BLMS (2021), S.P.)

Let $p \geq 1$. If C_ϕ is an isometry with respect to the norm $\|\cdot\|_{BMOA_p}$, then the set $\mathbb{D} \setminus \phi(\mathbb{D})$ has zero logarithmic capacity.

From the Hardy-Stein Identity,

$$\begin{aligned}\|f \circ \sigma_a - f(a)\|_p^p &= \frac{p^2}{2\pi} \int_{\mathbb{D}} |f(z) - f(a)|^{p-2} |f'(z)|^2 \log \left| \frac{1 - \bar{a}z}{a - z} \right| dA(z) \\ &=: I_p(f, a).\end{aligned}$$

Let

$$B_p(f) = \sup_{a \in \mathbb{D}} (I_p(f, a))^{1/p}.$$

Lemma

Let $p \in [1, 2)$. For every $w \in \mathbb{D}$ and for every $a \in \mathbb{D} \setminus \{w\}$,

$$B_p(\sigma_w) = 1 = (I_p(\sigma_w, w))^{1/p} > (I_p(\sigma_w, a))^{1/p}.$$

Also,

$$\lim_{|a| \rightarrow 1} I_p(\sigma_w, a) = 0.$$

Denote

$$H_\phi(w, z) = \log \left| \frac{1 - \bar{w}\phi(z)}{w - \phi(z)} \right| - N_\phi(w, z), \quad w, z \in \mathbb{D},$$

$H_\phi \geq 0$ on $\mathbb{D} \times \mathbb{D}$ with equality if and only if ϕ is an inner function.

$$\begin{aligned}
& I_p(C_\phi(g), a) \\
&= \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(\phi(z)) - g(\phi(a))|^{p-2} |(g \circ \phi)'(z)|^2 \log \left| \frac{1 - \bar{a}z}{a - z} \right| dA(z) \\
&= \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(\phi(z)) - g(\phi(a))|^{p-2} |g'(\phi(z))|^2 |\phi'(z)|^2 \log \left| \frac{1 - \bar{a}z}{a - z} \right| dA(z) \\
&= \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(w) - g(\phi(a))|^{p-2} |g'(w)|^2 N_\phi(w, a) dA(w) \\
&= \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(w) - g(\phi(a))|^{p-2} |g'(w)|^2 \log \left| \frac{1 - \bar{w}\phi(a)}{w - \phi(a)} \right| dA(w) \\
&\quad - \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(w) - g(\phi(a))|^{p-2} |g'(w)|^2 H_\phi(w, a) dA(w) \\
&= I_p(g, \phi(a)) - \frac{p^2}{2\pi} \int_{\mathbb{D}} |g(w) - g(\phi(a))|^{p-2} |g'(w)|^2 H_\phi(w, a) dA(w).
\end{aligned}$$

(\implies) Suppose that C_ϕ is an isometry with respect to the norm $\|\cdot\|_{BMOA_p}$.

We will consider two cases depending on whether the supremum in B_p is attained in \mathbb{D} or in the boundary of \mathbb{D} .

Case 1: For every $g \in BMOA$,

$$B_p(C_\phi(g)) = \lim_{n \rightarrow \infty} (I_p(C_\phi(g), a_n))^{1/p} \implies \lim_{n \rightarrow \infty} |a_n| = 1.$$

Let $b \in \mathbb{D}$. Let $g = \sigma_b$ and let (a_n) be a sequence in \mathbb{D} for which

$$B_p(C_\phi(\sigma_b)) = \lim_{n \rightarrow \infty} (I_p(C_\phi(\sigma_b), a_n))^{1/p}$$

$$\begin{aligned}
& (B_p(C_\phi(\sigma_b)))^p \\
= & \lim_{n \rightarrow \infty} I_p(C_\phi(\sigma_b), a_n) \\
= & \lim_{n \rightarrow \infty} \left(I_p(\sigma_b, \phi(a_n)) \right. \\
& \quad \left. - \frac{p^2}{2\pi} \int_{\mathbb{D}} |\sigma_b(w) - \sigma_b(\phi(a_n))|^{p-2} |\sigma'_b(w)|^2 H_\phi(w, a_n) dA(w) \right) \\
\leq & \limsup_{n \rightarrow \infty} I_p(\sigma_b, \phi(a_n)) \\
\leq & (B_p(\sigma_b))^p \\
= & (B_p(C_\phi(\sigma_b)))^p
\end{aligned}$$

$$(B_p(\sigma_b))^p = \lim_{n \rightarrow \infty} I_p(\sigma_b, \phi(a_n)) \implies \lim_{n \rightarrow \infty} \phi(a_n) = b \quad (\text{Lemma})$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{D}} |\sigma_b(w) - \sigma_b(\phi(a_n))|^{p-2} |\sigma'_b(w)|^2 H_\phi(w, a_n) dA(w) \right) &= 0. \\ \implies \lim_{n \rightarrow \infty} H_\phi(w, a_n) &= 0, \end{aligned}$$

for almost every $w \in \mathbb{D}$.

Case 2: There exist $g \in BMOA$ and $a \in \mathbb{D}$ such that

$$B_p(C_\phi(g)) = (I_p(C_\phi(g), a))^{1/p}.$$

Similarly, $H_\phi = 0 \implies \phi$ is inner.

(\Leftarrow) Since $B_p(C_\phi(f)) \leq B_p(f)$, it remains to prove that $B_p(C_\phi(f)) \geq B_p(f)$ for every $f \in BMOA$.

Let $g \in BMOA$ and $\epsilon > 0$. There exists $b \in \mathbb{D}$ such that

$$(B_p(g))^{1/p} \leq I_p(g, b) + \epsilon.$$

Let (a_n) be a sequence in \mathbb{D} such that

$$\lim_{n \rightarrow \infty} \phi(a_n) = b$$

and

$$\lim_{n \rightarrow \infty} H_\phi(w, a_n) = 0,$$

for almost every $w \in \mathbb{D}$.

Then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D}} |g(w) - g(\phi(a_n))|^{p-2} |g'(w)|^2 H_\phi(w, a_n) dA(w) = 0$$

and

$$\lim_{n \rightarrow \infty} I_p(C_\phi(g), a_n) = \lim_{n \rightarrow \infty} I_p(g, \phi(a_n)) = I_p(g, b).$$

We conclude that

$$(B_p(g))^{1/p} \leq I_p(g, b) + \epsilon = \lim_{n \rightarrow \infty} I_p(C_\phi(g), a_n) + \epsilon \leq (B_p(C_\phi(g)))^{1/p} + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $B_p(C_\phi(g)) \geq B_p(g)$.

Theorem (BLMS (2021), S.P.)

Let $p \geq 1$. If C_ϕ is an isometry with respect to the norm $\|\cdot\|_{BMOA_p}$, then the set $\mathbb{D} \setminus \phi(\mathbb{D})$ has zero logarithmic capacity.

Suppose that the set $\mathbb{D} \setminus \phi(\mathbb{D})$ has positive logarithmic capacity. Then

$$U(w, z) := G_{\mathbb{D}}(w, z) - G_{\phi(\mathbb{D})}(w, z) > 0, \quad (1)$$

Let (a_n) be a sequence in \mathbb{D} such that

$$B_p(C_\phi(\sigma_0)) = \lim_{n \rightarrow \infty} (I_p(C_\phi(\sigma_0), a_n))^{1/p}$$

and $(\phi(a_n))$ converges.

From the Lindelöf Principle,

$$N_\phi(w, a_n) \leq G_{\phi(\mathbb{D})}(w, \phi(a_n)) = \log \left| \frac{1 - \overline{\phi(a_n)}w}{\phi(a_n) - w} \right| - U(w, \phi(a_n)),$$

for all $w \in \mathbb{D}$ and $n \in \mathbb{N}$.

$$\begin{aligned}
& (B_p(C_\phi(\sigma_0)))^p \\
= & \lim_{n \rightarrow \infty} I_p(C_\phi(\sigma_0), a_n) \\
= & \lim_{n \rightarrow \infty} \left(\frac{p^2}{2\pi} \int_{\mathbb{D}} |w - \phi(a_n)|^{p-2} N_\phi(w, a_n) dA(w) \right) \\
\leq & \lim_{n \rightarrow \infty} \left(\frac{p^2}{2\pi} \int_{\mathbb{D}} |w - \phi(a_n)|^{p-2} \log \left| \frac{1 - \overline{\phi(a_n)}w}{\phi(a_n) - w} \right| dA(w) \right. \\
& \quad \left. - \frac{p^2}{2\pi} \int_{\mathbb{D}} |w - \phi(a_n)|^{p-2} U(w, \phi(a_n)) dA(w) \right) \\
= & \lim_{n \rightarrow \infty} \left(I_p(\sigma_0, \phi(a_n)) \right. \\
& \quad \left. - \frac{p^2}{2\pi} \int_{\mathbb{D}} |w - \phi(a_n)|^{p-2} U(w, \phi(a_n)) dA(w) \right).
\end{aligned}$$

Assuming that

$$\left| \lim_{n \rightarrow \infty} \phi(a_n) \right| = 1,$$

it follows from the Lemma that $B_p(C_\phi(\sigma_0)) \leq 0$, which implies that ϕ is constant, contradicting our assumption. Therefore the sequence $(\phi(a_n))$ converges to a point $b \in \mathbb{D}$.

$$\begin{aligned} & (B_p(C_\phi(\sigma_0)))^p \\ \leq & I_p(\sigma_0, b) - \frac{p^2}{2\pi} \int_{\mathbb{D}} |w - b|^{p-2} U(w, b) dA(w) \\ \leq & (B_p(\sigma_0))^p - \frac{p^2}{2\pi} \int_{\mathbb{D}} |w - b|^{p-2} U(w, b) dA(w) \\ < & (B_p(\sigma_0))^p, \end{aligned}$$

so C_ϕ is not an isometry of $BMOA$ for $\|\cdot\|_{BMOA_p}$.

Thank you!