A decomposition of analytic function spaces, with applications

Jonathan R. Partington (Leeds, UK)

Fields Institute, July 2021

Joint work with Eva Gallardo-Gutiérrez
with additional material from
Isabelle Chalendar and Daniel Seco

Hilbert function spaces on the disc

For $\alpha \in \mathbb{R}$, we write $\mathcal{A}_\alpha$ for the space of analytic functions in the unit disc $\mathbb{D}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$\|f\|^2 = \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty.$$

So $\mathcal{A}_0 = H^2$ (Hardy space), $\mathcal{A}_1 = \mathcal{D}$ (Dirichlet space), and $\mathcal{A}_{-1} = A^2$ (Bergman space).

An integral representation

For non-negative integers $N > \alpha/2$, A_α consists of all functions f such that

$$\int_{\mathbb{D}} |f^{(N)}(z)|^2 (1 - |z|^2)^{2N-\alpha-1}\, dA(z) < \infty$$

the integral taken with respect to Lebesgue area measure.

So for the Dirichlet space we recover the familiar condition

$$\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty.$$

Blaschke products and model spaces

Let $z_1, \ldots, z_n \in \mathbb{D}$ (not necessarily distinct). Then

$$B(z) := \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}$$

is a **finite Blaschke product**. It maps $\mathbb{D}$ to $\mathbb{D}$ and $|B(e^{it})| = 1$ for each $t \in \mathbb{R}$.

We write $K_B = H^2 \ominus BH^2$, the **model space**, an n-dimensional space of rational functions which is invariant under the backward shift

$$S^* f(z) = (f(z) - f(0))/z.$$

A decomposition of H^2

It is well known, and easy to prove, that

$$H^2 = K_B \oplus BK_B \oplus B^2K_B \oplus \dots$$

as an orthogonal decomposition. Note that T_B (multiplication by B) is an isometry on H^2.

We may write each $f \in H^2$ as $f = \sum_{r=0}^{\infty} h_r B^r$ with $h_r \in K_B$ and

$$\|f\|^2 = \sum_{r=0}^{\infty} \|h_r\|^2.$$

For $B(z) = z$, $K_B = \mathbb{C}$, the h_r are constants and this is the usual power series for f.

A decomposition for $\mathcal{A}_\alpha$

The analogous result is that in a precise sense

$$\mathcal{A}_\alpha = K_B \oplus BK_B \oplus B^2 K_B \oplus \dots$$

as before. This is NOT an orthogonal decomposition in general, but $f \in \mathcal{A}_\alpha$ if and only if

$$f = \sum_{r=0}^{\infty} h_r B^r$$

with $h_r \in K_B$ and $\sum_{r=0}^{\infty} (r+1)^\alpha \|h_r\|^2 < \infty$.

Note that since $\dim K_B < \infty$ it doesn't matter which (fixed) norm on K_B we take, as all are equivalent.

More on the decomposition

We have $f \in \mathcal{A}_\alpha$ if and only if

$$f = \sum_{r=0}^{\infty} h_r B^r$$

with $h_r \in K_B$ and $\sum_{r=0}^{\infty}(r+1)^\alpha \|h_r\|^2 < \infty$.
The result is obvious for $B(z) = z$, and in that special case the (power series) decompositions are orthogonal.

For any finite Blaschke product B we have a norm equivalence:

$$\left\| \sum_{r=0}^{\infty} h_r B^r \right\|_{A_\alpha}^2 \approx \sum_{r=0}^{\infty} (r+1)^\alpha \|h_r\|^2.$$

Some history and proofs

This was originally done in [CGP15] for Bergman and Dirichlet, then extended in [GPS20] to $\alpha \in [-1, 1]$, and finally to all α in [GP21].

Two basic facts are used in the proof:
1. The composition operator $C_B : f \mapsto f \circ B$ is bounded on $\mathcal{A}_\alpha$.
2. The Toeplitz (multiplication) operator $T_B : f \mapsto Bf$ is bounded on $\mathcal{A}_\alpha$.

These facts can be deduced fairly easily from the integral representation of $\mathcal{A}_\alpha$.

Application 1: weighted composition operators

Let $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic and $u \in \mathrm{Hol}(\mathbb{D})$. When is the weighted composition operator $W_{u,\varphi} : f \to u(f \circ \varphi)$ bounded on A_α?

One way to analyse this is via multiplier spaces. Let

$$\mathcal{M}_\alpha(\varphi) = \{u \in \mathrm{Hol}(\mathbb{D}) : W_{u,\varphi} \text{ is bounded on } A_\alpha\}.$$

For H^2, it is known that $M_0(\varphi) = H^\infty$ (as small as possible) if and only if φ is a finite Blaschke product (Contreras–Hernández-Díaz (2003) and others).

Weighted composition operators on $\mathcal{D}$: I

With the aid of the decomposition, a similar result can be proved for the Dirichlet space.

Let $\mathcal{M}(\mathcal{D})$ be the **multiplier space** of $\mathcal{D}$, i.e., all $u \in \mathrm{Hol}(\mathbb{D})$ such that $T_u : \mathcal{D} \to \mathcal{D}$ is bounded.

It is known that $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{D} \cap H^\infty$. Clearly, $M_1(\varphi) \supseteq M(\mathcal{D})$ if C_φ is bounded on $\mathbb{D}$.

Theorem. Let φ be an inner function. Then $\mathcal{M}_1(\varphi) = \mathcal{M}(\mathcal{D})$ if and only if φ is a finite Blaschke product.

However, there are functions φ with $\|\varphi\|_\infty = 1$ for which the multiplier space is larger.

Weighted composition operators on $\mathcal{D}$: II

At the other extreme we have the following result (which does not use the decomposition).

Theorem. Let φ be an analytic self-map of $\mathbb{D}$.
Then $\mathcal{M}_1(\varphi) = \mathcal{D}$ if and only if
(i) $\|\varphi\|_\infty < 1$, and
(ii) $\varphi \in \mathcal{M}(\mathcal{D})$.

These ideas were also used to analyse spectral properties of weighted composition operators (more technical, and omitted here).

Application 2: wandering subspaces

An operator T on a Hilbert space $\mathcal{H}$ has the **wandering subspace property** if every closed invariant subspace $\mathcal{M}$ is generated under T by $\mathcal{M} \ominus T\mathcal{M}$. We write this as

$$[\mathcal{M} \ominus T\mathcal{M}]_T = \mathcal{M}.$$

The Wold decomposition theorem for a shift (pure isometry)

$$\mathcal{H} = \mathcal{K} \oplus T\mathcal{K} \oplus T^2\mathcal{K} \oplus \dots$$

shows that this holds for these operators.

It is still not known exactly which multiplication operators, even on H^2, have the property.

Wandering subspaces for non-isometries

When it comes to other $\mathcal{A}_\alpha$ the situation, even for multiplication by z^k, is much more complicated. Even a clear description of T_z-invariant subspaces isn't available.

Work of Aleman–Richter–Sundberg and Shimorin shows that for T_z we do have the wandering subspace property for $\alpha \in [-1, 1]$.

Still unknown for T_{z^2} in general.
Moreover Seco has found α and k for which the wandering subspace property fails for z^k on $\mathcal{A}_\alpha$. For example, $\alpha = -16$ and $k = 6$.

Shimorin's theorem

Suppose that an operator T satisfies $\bigcap_{n=1}^{\infty} T^n H = \{0\}$ and one of the following two conditions:
(i) $\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2$ for all x (concavity);
(ii) $\|x + Ty\|^2 \leq 2(\|Tx\|^2 + \|y\|^2)$ for all x and y.

Then T has the wandering subspace property.

In particular (i) gives the property for T_z on $\mathcal{A}_\alpha$ for $\alpha \in [0, 1]$ (e.g. Dirichlet), and (ii) gives it for $\alpha \in [-1, 0]$.

Equivalent norms

Theorem. Let $\alpha \in [-1, 1]$ and B a finite Blaschke product. Then there exists an equivalent norm $\|.\|_B$ under which T_B has the wandering subspace property in $\mathcal{A}_\alpha$, that is, for any invariant subspace $\mathcal{M}$ we have $[\mathcal{M} \ominus T_B\mathcal{M}]_B = \mathcal{M}$.

Moreover, for $B(z) = z^k$ and $\alpha \in [0, \log(2)/\log(k+1)]$, the norm $\|.\|_B$ coincides with the usual norm on $\mathcal{A}_\alpha$.

Idea of proof. Use the decomposition given earlier to construct a norm under which T_B is unitarily equivalent to a shift acting on a vector-valued $\mathcal{A}_\alpha$. Then use the theorem of Shimorin.

Application 3: commutants of multiplication operators

Once again, a lot is known about Hardy spaces, and much less about other $\mathcal{A}_\alpha$.

In the 1970s and 1980s, Cowen, Thomson, Deddens, Wong, Shields, Wallen, etc. gave fairly complete results on the commutant of a multiplier T_g on H^2, with $g \in H^\infty$.

Some results on the Bergman space are available, e.g. Douglas–Sun–Zheng, Cuckovic, and very recently Abkar–Cao–Zhu (2020), who looked at multiplication by z^k.

Once again, we can say a lot more using the decomposition given earlier.

Rewriting the decomposition theorem

Let B be a finite Blaschke product, and $u_1, \ldots, u_n$ a basis for the model space K_B.

For $\alpha \in \mathbb{R}$ and $f \in \mathcal{A}_\alpha$ we have a unique decomposition

$$f = \sum_{j=1}^{n} u_j f_j(B),$$

with $f_1, \ldots, f_n \in \mathcal{A}_\alpha$. Moreover $\|f\|$ is equivalent to $\left(\sum_{j=1}^{n} \|f_j\|^2\right)^{1/2}$.

The commutant theorem

With the notation as above, a bounded linear operator W on $\mathcal{A}_\alpha$ commutes with T_B if and only if

$$W\left(\sum_{j=1}^{n} u_j f_j(B)\right) = \sum_{j=1}^{n} \varphi_j f_j(B)$$

for some $\varphi_1, \ldots, \varphi_n \in \mathcal{M}(\mathcal{A}_\alpha)$ (multipliers of $\mathcal{A}_\alpha$). For example, if $B(z) = z^n$, we can take $u_j = z^{j-1}$ for $j = 1, 2, \ldots, n$, and we have

$$W\left(\sum_{k=0}^{n-1} z^k f_k(z^n)\right) = \sum_{k=0}^{n-1} \varphi_k f_k(z^n),$$

which is basically the Abkar–Cao–Zhu result.

Reducing subspaces

Recall that a reducing subspace $\mathcal{M}$ of an operator T is one that is invariant under T and T^*, or equivalently both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under T.

Another way of saying this is that the orthogonal projection $P_{\mathcal{M}}$ commutes with both T and its adjoint T^*.

There has been much work on this, especially for T_B on the Bergman space. We mention Douglas, Putinar, Sun, Wang and their co-authors.

Matrix notation: I

First, the commutant of T_B is isomorphic to the space of $n \times n$ matrices of multipliers of A_α. With

$$W\left(\sum_{j=1}^{n} u_j f_j(B)\right) = \sum_{k=1}^{n} \varphi_k f_k(B)$$

and since $\varphi_k \in \mathcal{A}_\alpha$, we may write

$$\varphi_k = \sum_{j=1}^{n} u_j \varphi_{jk}(B), \qquad \text{so that}$$

$$W\left(\sum_{j=1}^{n} u_j f_j(B)\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} u_j \varphi_{jk}(B) f_k(B).$$

Matrix notation: II

Thus, identifying $\sum_{j=1}^{n} u_j f_j(B) \in \mathcal{A}_\alpha$ with the column vector $(f_1, \ldots, f_n)^T$ we have

$$W : \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \mapsto \Phi \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

where Φ is a matrix of multipliers of $\mathcal{A}_\alpha$.

This makes it easy to find projections in the commutant of T_B, and further work is possible (we have mainly explored the Hardy and Bergman spaces).

Finally - an orthogonal decomposition

For B a degree-n Blaschke product we easily see that every $B^k\mathcal{D}$ ($k \geq 1$) is closed. We may therefore write

$$\mathcal{D} = K_0 \oplus K_1 \oplus K_2 \oplus \dots$$

where $X_0 = \mathcal{D} \ominus B\mathcal{D}$, the Dirichlet "model space", and $X_k = B^k\mathcal{D} \ominus B^{k+1}\mathcal{D}$ for each $k \geq 1$.

With this decomposition, T_B is now lower triangular, and every self-adjoint W in its commutant is block diagonal.

Unfortunately, usable bases for the n-dim. subspaces X_k, $k \geq 1$, are unknown so far.

Some references

[CGP15] I. Chalendar, E.A. Gallardo-Gutiérrez and J.R. Partington, Weighted composition operators on the Dirichlet space: boundedness and spectral properties. *Math. Annalen* 363 (2015), 1265–1279.

[GP21] E.A. Gallardo-Gutiérrez and J.R. Partington, Multiplication by a finite Blaschke product on weighted Bergman spaces: commutant and reducing subspaces. https://arxiv.org/abs/2105.07760.

[GPS20] E.A. Gallardo-Gutiérrez, J.R. Partington and D. Seco, On the wandering property in Dirichlet spaces. *Integral Equations and Operator Theory* 92 (2020), paper no. 16.

That's all. Thank you.