Mini-course on the Dirichlet space

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Chapter 1

Introduction

What is the Dirichlet space?

The Dirichlet space $\mathcal{D}$

$\mathcal{D}$ is the set of f holomorphic in $\mathbb{D}$ whose Dirichlet integral is finite:

$$\mathcal{D}(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

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- If $f(z) = \sum_{k\geq 0} a_k z^k$, then $\mathcal{D}(f) = \sum_{k\geq 0} k|a_k|^2$. Consequently $\mathcal{D} \subset H^2$.

- $\mathcal{D}$ is a Hilbert space with respect to the norm $\|\cdot\|_{\mathcal{D}}$ given by

$$\|f\|_{\mathcal{D}}^2 := \|f\|_{H^2}^2 + \mathcal{D}(f) = \sum_{k\geq 0}(k+1)|a_k|^2.$$

History and motivation

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Some reasons for studying $\mathcal{D}$:

- Potential theory, energy, capacity
- Geometric interpretation, Möbius invariance
- Weighted shifts, invariant subspaces
- Borderline case, still many open problems

What to study?

Some topics of interest:

- Boundary behavior
- Zeros
- Multipliers
- Reproducing kernel
- Interpolation
- Conformal invariance
- Shift-invariant subspaces

Where to find out more about $\mathcal{D}$?

Survey articles:

- W. Ross, *The classical Dirichlet space*, Recent advances in operator-related function theory, 171–197, Contemp. Math., 393, Amer. Math. Soc., Providence, RI, 2006.
- N. Arcozzi, R. Rochberg, E. Sawyer, B. Wick, *The Dirichlet space: a survey*, New York J. Math. 17A (2011), 45–86.

Monographs:

- O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford, *A primer on the Dirichlet space*, Cambridge University Press, Cambridge, 2014
- N. Arcozzi, R. Rochberg, E. Sawyer, B. Wick, *The Dirichlet space and related function spaces*, Amer. Math. Soc., Providence RI, 2019.

Chapter 2

Capacity

Energy

Let μ be a finite positive Borel measure on $\mathbb{T}$.

Energy of μ

$$I(\mu) := \int_{\mathbb{T}} \int_{\mathbb{T}} \log \frac{2}{|\lambda - \zeta|} \, d\mu(\lambda) \, d\mu(\zeta).$$

- May have $I(\mu) = +\infty$.
- Formula for $I(\mu)$ in terms of Fourier coefficients of μ:

$$I(\mu) = \sum_{k \geq 1} \frac{|\widehat{\mu}(k)|^2}{k} + \mu(\mathbb{T})^2 \log 2.$$

Capacity of compact sets

Capacity of compact $F \subset \mathbb{T}$

$$c(F) := 1/\inf\{I(\mu) : \ \mu \text{ is a probability measure on } F\}.$$

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Elementary properties:

- $F_1 \subset F_2 \Rightarrow c(F_1) \leq c(F_2)$
- $F_n \downarrow F \Rightarrow c(F_n) \downarrow c(F)$
- $c(F_1 \cup F_2) \leq c(F_1) + c(F_2)$

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Examples:

- $c(F) \leq 1/\log(2/\operatorname{diam}(F))$
- $c(F) = 0$ if F is finite or countable
- $c(F) \geq 1/\log(2\pi e/|F|)$. In particular $c(F) = 0 \Rightarrow |F| = 0$.
- $c(F) > 0$ if F is the (circular) middle-third Cantor set.

Capacity of general sets

Inner capacity of $E \subset \mathbb{T}$

$$c(E) := \sup\{c(F) : \text{compact } F \subset E\}$$

Outer capacity of $E \subset \mathbb{T}$

$$c^*(E) := \inf\{c(U) : \text{open } U \supset E\}$$

- $c^*(\cup_n E_n) \leq \sum_n c^*(E_n)$ (not true for $c(\cdot)$).
- $c^*(E) = c(E)$ if E is Borel (Choquet's capacitability theorem)
- A property holds q.e. if it holds outside an E with $c^*(E) = 0$.

Equilibrium measures

Let F be a compact subset of $\mathbb{T}$. Recall that

$$c(F) := 1/\inf\{I(\mu) : \ \mu \text{ is a probability measure on } F\}.$$

Measure μ attaining the inf is called an *equilibrium measure* for F.

Proposition

If $c(F) > 0$, then F admits a unique equilibrium measure.

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Fundamental theorem of potential theory (Frostman, 1935)

Let μ be the equilibrium measure for F, and V_μ be its potential, i.e.

$$V_\mu(z) := \int_{\mathbb{T}} \log \frac{2}{|z-\zeta|}\, d\mu(\zeta).$$

Then $V_\mu \leq 1/c(F)$ on $\mathbb{T}$, and $V_\mu = 1/c(F)$ q.e. on F.

Chapter 3

Boundary behavior

Preliminary remarks

- Every $f \in \mathcal{D}$ has non-tangential limits a.e. on $\mathbb{T}$ (as $f \in H^2$).
- There exists $f \in \mathcal{D}$ such that $\lim_{r \to 1^-} |f(r)| = \infty$.

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Example: Consider

$$f(z) := \sum_{k \geq 2} \frac{z^k}{k \log k}.$$

Then

$$\mathcal{D}(f) = \sum_{k \geq 2} k \frac{1}{(k \log k)^2} = \sum_{k \geq 2} \frac{1}{k(\log k)^2} < \infty,$$

but

$$\liminf_{r \to 1^-} f(r) \geq \sum_{k \geq 2} \frac{1}{k \log k} = \infty.$$

Beurling's theorem

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If $f \in \mathcal{D}$ then f has non-tangential limits q.e. on $\mathbb{T}$.

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Remarks:

- Beurling actually proved his result just for radial limits
- Beurling's theorem is sharp in the following sense:

Theorem (Carleson, 1952)

Given compact $E \subset \mathbb{T}$ of capacity zero, there exists $f \in \mathcal{D}$ such that $\lim_{r\to 1^-} |f(r\zeta)| = \infty$ for all $\zeta \in E$.

Capacitary weak-type and strong-type inequalities

Notation: Let $f \in \mathcal{D}$. For $\zeta \in \mathbb{T}$, we write $f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$.
Also A, B denote absolute positive constants.

Weak-type inequality (Beurling, 1940)

$$c(|f^*| > t) \le A\|f\|_{\mathcal{D}}^2/t^2 \quad (t > 0).$$

Corollary

$$|\{|f^*| > t\}| \le Ae^{-Bt^2/\|f\|_{\mathcal{D}}^2} \quad (t > 0).$$

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Strong-type inequality (Hansson, 1979)

$$\int_0^\infty c(|f^*| > t)\, t\, dt \leq A\|f\|_{\mathcal{D}}^2.$$

Douglas' formula

Theorem (Douglas, 1931)

If $f \in H^2$, then

$$\mathcal{D}(f) = \frac{1}{4\pi^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f^*(\lambda) - f^*(\zeta)}{\lambda - \zeta} \right|^2 |d\lambda|\,|d\zeta|.$$

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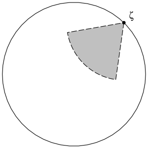
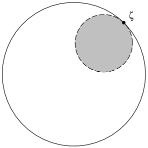
Corollary

If $f \in \mathcal{D}$, then f has oricyclic limits a.e. in $\mathbb{T}$.



non-tangential approach region



oricyclic approach region

Exponential approach region

Theorem (Nagel–Rudin–Shapiro, 1982)

If $f \in \mathcal{D}$ then, for a.e. $\zeta \in \mathbb{D}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in the exponential approach region

$$|z - \zeta| < \kappa\Big(\log \frac{1}{1 - |z|}\Big)^{-1}.$$

Remarks:

- Approach region is 'widest possible'.
- This is an a.e. result (not q.e.).

Carleson's formula

Notation: Let $f \in H^2$ with canonical factorization $f = BSO$.
Let (a_n) be the zeros of B, and σ be the singular measure of S.

Theorem (Carleson, 1960)

$$\mathcal{D}(f) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(|f^*(\lambda)|^2 - |f^*(\zeta)|^2)(\log|f^*(\lambda)| - \log|f^*(\zeta)|)}{|\lambda-\zeta|^2}\,\frac{|d\lambda|}{2\pi}\,\frac{|d\zeta|}{2\pi}$$
$$+ \int_{\mathbb{T}}\Big(\sum_n \frac{1-|a_n|^2}{|\zeta-a_n|^2} + \int_{\mathbb{T}} \frac{2}{|\lambda-\zeta|^2}\,d\sigma(\lambda)\Big)|f^*(\zeta)|^2\,\frac{|d\zeta|}{2\pi}.$$

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Corollary 1

If f belongs to $\mathcal{D}$ then so does its outer factor.

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Corollary 1

If f belongs to $\mathcal{D}$ then so does its outer factor.

Corollary 2

The only inner functions in $\mathcal{D}$ are finite Blaschke products.

Some further developments

- Chang–Marshall theorem (1985):

$$\sup\Big\{\int_{\mathbb{T}} \exp(|f^*(e^{i\theta})|^2)\, d\theta : f(0)=0,\ \mathcal{D}(f)\le 1\Big\} < \infty.$$

- Trade-off between approach regions and exceptional sets. Borichev (1994), Twomey (2002)

Chapter 4

Zeros

Preliminary remarks

A sequence (z_n) in $\mathbb{D}$ (possibly with repetitions) is:

- a *zero set* for $\mathcal{D}$ if $\exists f \in \mathcal{D}$ vanishing on (z_n) but $f \not\equiv 0$;
- a *uniqueness set* for $\mathcal{D}$ if it is not a zero set.

Proposition

If (z_n) is a zero set for $\mathcal{D}$, then $\exists f \in \mathcal{D}$ vanishing precisely on (z_n).

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It is well known that (z_n) is a zero set for the Hardy space H^2 iff

$$\sum_n (1 - |z_n|) < \infty.$$

What about the Dirichlet space?

The three cases

Case I (obvious)

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Case III (Nagel–Rudin–Shapiro, 1982)

If (z_n) *satisfies neither condition, then there exist a zero set* (z_n') *and a uniqueness set* (z_n'') *with* $|z_n| = |z_n'| = |z_n''|$ *for all* n.

Thus, in Case **III**, the arguments of (z_n) matter. Back to this later.

Boundary zero sets

Let E be a closed subset of $\mathbb{T}$. It is called a *Carleson set* if

$$\int_{\mathbb{T}} \log\left(\frac{2}{\operatorname{dist}(\zeta, E)}\right) |d\zeta| < \infty.$$

Theorem (Carleson 1952)

If E is a Carleson set, then $\exists f \in A^1(\mathbb{D})$ with $f^{-1}(0) = E$.

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If $c(E) = 0$, then $\exists f \in \mathcal{D} \cap A(\mathbb{D})$ with $f^{-1}(0) = E$.

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- Neither result implies the other.
- Clearly, if $|E| > 0$, then E is a boundary uniqueness set for $\mathcal{D}$. But there also exist closed uniqueness sets E with $|E| = 0$.

Arguments of zero sets

We return to zero sets within $\mathbb{D}$, now considering their arguments.

Theorem (Caughran, 1970)

Let $(e^{i\theta_n})$ be a sequence in $\mathbb{T}$. The following are equivalent:

- $(r_n e^{i\theta_n})$ *is a zero set for* $\mathcal{D}$ *whenever* $\sum_n (1 - r_n) < \infty$.
- $E := \overline{\{e^{i\theta_n} : n \geq 1\}}$ *is a Carleson set.*

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Example of a Blaschke sequence that is a uniqueness set for $\mathcal{D}$

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There is still no satisfactory complete characterization of zero sets.

Some further developments

- Carleson sets as zero sets for $A^\infty(\mathbb{D})$
  Taylor–Williams (1970)

Chapter 5

Multipliers

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- $f \mapsto f(z)$ is a character, so $|f(z)| \leq$ spectral radius of f.

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Proof: Suppose $\mathcal{D}$ is an algebra.

- By closed graph theorem, $\mathcal{D}$ isomorphic to a Banach algebra.
- $f \mapsto f(z)$ is a character, so $|f(z)| \leq$ spectral radius of f.
- Therefore every $f \in \mathcal{D}$ is bounded. Contradiction.

Multipliers

Definition

A *multiplier* for $\mathcal{D}$ is a function ϕ such that $\phi f \in \mathcal{D}$ for all $f \in \mathcal{D}$. The set of multipliers is an algebra, denoted by $\mathcal{M}(\mathcal{D})$.

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When is ϕ a multiplier of $\mathcal{D}$?

- Necessary condition: $\phi \in \mathcal{D} \cap H^\infty$
- Sufficient condition: $\phi' \in H^\infty$

To completely characterize multipliers, we introduce a new notion.

Carleson measures

Definition

A measure μ on $\mathbb{D}$ is a *Carleson measure* for $\mathcal{D}$ if $\exists C$ such that

$$\int_{\mathbb{D}} |f|^2 \, d\mu \leq C \|f\|_{\mathcal{D}}^2 \quad (f \in \mathcal{D}).$$

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With this notion in hand, it is quite easy to characterize multipliers:

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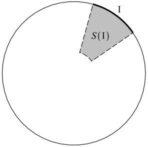
Begs a new question: how to characterize Carleson measures?

Characterization of Carleson measures



Let μ be a finite positive measure on $\mathbb{D}$.
$S(I) := \{re^{i\theta} : 1 - |I| < r < 1,\ e^{i\theta} \in I\}$.
Carleson (1962): μ is Carleson for H^2 iff $\mu(S(I)) = O(|I|)$.

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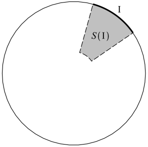
Characterization of Carleson measures



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When is μ a Carleson measure for $\mathcal{D}$?

Theorem (Wynn, 2011)

The condition $\mu(S(I)) = O(\psi(|I|))$ is:

- *necessary if $\psi(x) := 1/\log(1/x)$;*
- *sufficient if $\psi(x) := 1/\log(1/x)(\log\log(1/x))^{\alpha}$ with $\alpha > 1$.*

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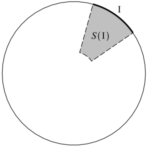

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Theorem (Stegenga, 1980)

μ is a Carleson measure for $\mathcal{D}$ iff there is a constant A such that, for every finite set of disjoint closed subarcs $I_1, \dots, I_n$ of $\mathbb{T}$,

$$\mu\Big(\cup_{j=1}^n S(I_j)\Big) \le Ac\Big(\cup_{j=1}^n I_j\Big).$$

Multipliers and reproducing kernels

If $f \in \mathcal{D}$ and $w \in \mathbb{D}$, then $f(w) = \langle f, k_w \rangle_{\mathcal{D}}$, where

$$k_w(z) := \frac{1}{\overline{w}z} \log\Big(\frac{1}{1 - \overline{w}z}\Big) \quad (w, z \in \mathbb{D}).$$

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Proposition

Let $\phi \in \mathcal{M}(\mathcal{D})$ and define $M_\phi : \mathcal{D} \to \mathcal{D}$ by $M_\phi(f) := \phi f$. Then

$$M_\phi^*(k_w) = \overline{\phi(w)} k_w \quad (w \in \mathbb{D}).$$

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Proof: For all $f \in \mathcal{D}$, we have

$$\langle f, M_\phi^*(k_w) \rangle_{\mathcal{D}} = \langle \phi f, k_w \rangle_{\mathcal{D}} = \phi(w) f(w) = \phi(w) \langle f, k_w \rangle_{\mathcal{D}} = \langle f, \overline{\phi(w)} k_w \rangle_{\mathcal{D}}.$$

Pick interpolation

Problem: Given $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \overline{\mathbb{D}}$, does there exist $\phi \in \mathcal{M}(\mathcal{D})$ with $\|M_\phi\| \leq 1$ such that $\phi(z_j) = w_j$ for all j?

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ϕ exists iff the matrix $(1 - \overline{w}_i w_j)\langle k_{z_i}, k_{z_j} \rangle_{\mathcal{D}}$ is positive semi-definite.

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- Necessity is a simple consequence of the preceding proposition. The same argument works for any RKHS.
- Sufficiency is a property of the Dirichlet kernel ('Pick property').

Interpolating sequences

A sequence $(z_n)_{n\geq 1}$ in $\mathbb{D}$ is an *interpolating sequence* for $\mathcal{M}(\mathcal{D})$ if

$$\Big\{(\phi(z_1), \phi(z_2), \phi(z_3), \dots) : \phi \in \mathcal{M}(\mathcal{D})\Big\} = \ell^\infty.$$

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Theorem (Marshall–Sundberg (1990's), Bishop (1990's), Bøe (2005))

The following are equivalent:

- $(z_n)_{n\geq 1}$ *is an interpolating sequence for* $\mathcal{M}(\mathcal{D})$;
- $\displaystyle\sum_n \frac{\delta_{z_n}}{\|k_{z_n}\|^2}$ *is a* $\mathcal{D}$*-Carleson measure and* $\displaystyle\sup_{\substack{n,m \\ n\neq m}} \frac{|\langle k_{z_n}, k_{z_m}\rangle_{\mathcal{D}}|}{\|k_{z_n}\|_{\mathcal{D}}\|k_{z_m}\|_{\mathcal{D}}} < 1.$

Factorization theorems

We say f is *cyclic* for $\mathcal{D}$ if $\overline{\mathcal{M}(\mathcal{D})f} = \mathcal{D}$.

- Clearly f cyclic $\Rightarrow f(z) \neq 0$ for all $z \in \mathbb{D}$. The converse is false.
- f is cyclic for H^2 iff f is an outer function (Beurling).

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If $f \in \mathcal{D}$, then $f = \phi g$, where $\phi \in \mathcal{M}(\mathcal{D})$ and g is cyclic in $\mathcal{D}$.

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Smirnov factorization (Aleman–Hartz–McCarthy–Richter, 2017)

If $f \in \mathcal{D}$, then $f = \phi_1/\phi_2$, where $\phi_1, \phi_2 \in \mathcal{M}(\mathcal{D})$ and ϕ_2 is cyclic in $\mathcal{D}$.

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Corollary

Given $f \in \mathcal{D}$, there exists $\phi \in \mathcal{M}(\mathcal{D})$ with the same zero set.
Consequently, the union of two zero sets is again one.

Some further developments

- Further characterizations of multipliers and Carleson measures for $\mathcal{D}$
  Arcozzi–Rochberg–Sawyer (2002)
- Reverse Carleson measures
  Fricain–Hartmann–Ross (2017)
- Corona problem for $\mathcal{M}(\mathcal{D})$
  Tolokonnikov (1991), Xiao (1998), Trent (2004)

Chapter 6

Conformal invariance

Preliminary remarks

Let $\phi : \mathbb{D} \to \mathbb{C}$ and $f : \phi(\mathbb{D}) \to \mathbb{C}$ be holomorphic functions.
Write $n_\phi(w)$ for the number of solutions z of $\phi(z) = w$.

Change-of-variable formula

$$\mathcal{D}(f \circ \phi) = \frac{1}{\pi} \int_{\phi(\mathbb{D})} |f'(w)|^2 n_\phi(w)\, dA(w).$$

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If ϕ is injective, then $\mathcal{D}(\phi) = (\text{area of } \phi(\mathbb{D}))/\pi$.

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Corollary 1

If ϕ is injective, then $\mathcal{D}(\phi) = (\text{area of } \phi(\mathbb{D}))/\pi$.

Corollary 2

If $f \in \mathcal{D}$ and $\phi \in \operatorname{aut}(\mathbb{D})$, then $f \circ \phi \in \mathcal{D}$ and $\mathcal{D}(f \circ \phi) = \mathcal{D}(f)$.

This last property more-or-less characterizes $\mathcal{D}$.

Characterization of $\mathcal{D}$ via Möbius invariance

Notation:

- $\mathcal{H}$:= a vector space of holomorphic functions on $\mathbb{D}$
- $\langle \cdot, \cdot \rangle$:= a semi-inner product on $\mathcal{H}$ and $\mathcal{E}(f) := \langle f, f \rangle$.

Theorem (Arazy–Fisher 1985, slightly modified)

Assume:

- *if $f \in \mathcal{H}$ and $\phi \in \mathrm{aut}(\mathbb{D})$, then $f \circ \phi \in \mathcal{H}$ and $\mathcal{E}(f \circ \phi) = \mathcal{E}(f)$;*
- *$\|f\|^2 := |f(0)|^2 + \mathcal{E}(f)$ defines a Hilbert-space norm on $\mathcal{H}$;*
- *convergence in this norm implies pointwise convergence on $\mathbb{D}$;*
- *$\mathcal{H}$ contains a non-constant function.*

Then $\mathcal{H} = \mathcal{D}$ and $\mathcal{E}(\cdot) \equiv a\mathcal{D}(\cdot)$ some constant $a > 0$.

Composition operators

Given holomorphic $\phi : \mathbb{D} \to \mathbb{D}$, define $C_\phi : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ by

$$C_\phi(f) := f \circ \phi.$$

If $\phi \in \mathrm{aut}(\mathbb{D})$ then $C_\phi : \mathcal{D} \to \mathcal{D}$. For which other ϕ is this true?

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If $\phi(z) := \sum_{k\geq 1} 2^{-k} z^{4^k}$, then $\phi : \mathbb{D} \to \mathbb{D}$, but $C_\phi(\mathcal{D}) \not\subset \mathcal{D}$ as $\phi \notin \mathcal{D}$.

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Theorem (MacCluer–Shapiro, 1986)

$$C_\phi : \mathcal{D} \to \mathcal{D} \iff \int_{S(I)} n_\phi \, dA = O(|I|^2).$$

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Corollary (El-Fallah–Kellay–Shabankhah–Youssfi, 2011)

Conditions for $C_\phi : \mathcal{D} \to \mathcal{D}$:

- *necessary:* $\mathcal{D}(\phi^k) = O(k)$ *as* $k \to \infty$.
- *sufficient:* $\mathcal{D}(\phi^k) = O(1)$ *as* $k \to \infty$.

Weighted composition operators

Theorem (Mashreghi–J. Ransford–T. Ransford, 2018)

Let $T : \mathcal{D} \to \mathrm{Hol}(\mathbb{D})$ *be a linear map. The following are equivalent:*

- T *maps nowhere-vanishing functions to nowhere-vanishing functions.*
- $\exists$ *holomorphic functions* $\phi : \mathbb{D} \to \mathbb{D}$ *and* $\psi : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ *such that*

$$Tf = \psi.(f \circ \phi) \quad (f \in \mathcal{D}).$$

Some further developments

- Compact composition operators on $\mathcal{D}$
  MacCluer, Shapiro (1986)

- Composition operators in Schatten classes
  Lefèvre, Li, Queffélec, Rodríguez-Piazza (2013)

- Geometry of $\phi(\mathbb{D})$ when C_ϕ is Hilbert–Schmidt
  Gallardo-Gutiérrez, Gonzalez (2003)

Chapter 7

Weighted Dirichlet spaces

The $\mathcal{D}_\alpha$ spaces

Definition

For $-1 < \alpha \leq 1$, write $\mathcal{D}_\alpha$ for the set of holomorphic f on $\mathbb{D}$ with

$$\mathcal{D}_\alpha(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \, dA(z) < \infty.$$

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Properties:

- $\mathcal{D}_\alpha(\sum_k a_k z^k) \asymp \sum_k k^{1-\alpha} |a_k|^2$
- $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 \cong H^2$
- If $0 < \alpha < 1$, then $\mathcal{D}_\alpha$ is 'akin' to $\mathcal{D}$ (using Riesz capacity c_α).
- If $-1 < \alpha < 0$, then $\mathcal{D}_\alpha$ is a subalgebra of the disk algebra.

The $\mathcal{D}_\mu$ spaces

Given a finite positive measure μ on $\mathbb{T}$, write $P\mu$ for its Poisson integral:

$$P\mu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta) \quad (z \in \mathbb{D}).$$

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Given μ, we denote by $\mathcal{D}_\mu$ the set of holomorphic f on $\mathbb{D}$ such that

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- If $\mu = d\theta/2\pi$, then $\mathcal{D}_\mu = \mathcal{D}$, the classical Dirichlet space.
- If $\mu = \delta_\zeta$, then $\mathcal{D}_\mu$ is the *local Dirichlet space* at ζ, denoted $\mathcal{D}_\zeta$.

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Note: Can recover $\mathcal{D}_\mu(f)$ from $\mathcal{D}_\zeta(f)$ using Fubini's theorem:

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\zeta(f)\, d\mu(\zeta).$$

Properties of $\mathcal{D}_\mu$ (Richter–Sundberg, 1991)

- $\mathcal{D}_\mu \subset H^2$ and is Hilbert space w.r.t. $\|f\|_{\mathcal{D}_\mu}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f)$.

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- Douglas formula: if $f \in \mathcal{D}_\mu$, then f^* exists μ-a.e. and

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- Carleson formula for $\mathcal{D}_\mu(f)$.
- Polynomials are dense in $\mathcal{D}_\mu$.
- $\mathcal{D}_\mu(f_r) \leq 4\mathcal{D}_\mu(f)$ (where $f_r(z) := f(rz)$).
  Can replace 4 by 1 (Sarason 1997, using de Branges–Rovnyak spaces).

Some further developments

- Capacities for $\mathcal{D}_\mu$.
  Chacón (2011), Guillot (2012)
- Estimates for reproducing kernel and capacities in $\mathcal{D}_\mu$.
  El-Fallah, Elmadani, Kellay (2019)
- Superharmonic weights
  Aleman (1993)
- $\mathcal{D}_\mu$ has the complete Pick property
  Shimorin (2002)

Chapter 8

Shift-invariant subspaces

Preliminary remarks

Notation:

- T a bounded operator on a Hilbert space $\mathcal{H}$
- $\mathrm{Lat}(T, \mathcal{H}) :=$ the lattice of closed T-invariant subspaces of $\mathcal{H}$.
- $M_z :=$ the shift operator (multiplication by z).

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Theorem (Beurling, 1948)

If $\mathcal{M} \in \mathrm{Lat}(M_z, H^2) \setminus \{0\}$, then $\mathcal{M} = \theta H^2$ where θ is inner.

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Analogue for $\mathrm{Lat}(M_z, \mathcal{D})$?

The shift operator on $\mathcal{D}_\mu$

Write $(T, \mathcal{H}) := (M_z, \mathcal{D})$. Clearly:

(1) $\|T^2 f\|^2 - 2\|Tf\|^2 + \|f\|^2 = 0$ for all $f \in \mathcal{H}$.

(2) $\cap_{n\geq 0} T^n(\mathcal{H}) = \{0\}$.

(3) $\dim(\mathcal{H} \ominus T(\mathcal{H})) = 1$.

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It turns out that the same properties hold if $(T, \mathcal{H}) := (M_z, \mathcal{D}_\mu)$.
Conversely:

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Conversely:

Theorem (Richter, 1991)

Let T be an operator on a Hilbert space $\mathcal{H}$ satisfying (1),(2),(3). Then there exists a unique finite measure μ on $\mathbb{T}$ such that $(T, \mathcal{H})$ is unitarily equivalent to $(M_z, \mathcal{D}_\mu)$.

Invariant subspaces of $(M_z, \mathcal{D})$

Let $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D})$.

- Clearly $(M_z, \mathcal{M})$ satisfies properties (1),(2).
- If $\mathcal{M} \neq \{0\}$, then (3) also holds (Richter–Shields 1988).

Leads to:

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Leads to:

Theorem (Richter 1991, Richter–Sundberg 1992)

Let $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D})$ and let $\phi \in \mathcal{M} \ominus M_z(\mathcal{M})$ with $\phi \not\equiv 0$. Then:

- *ϕ is a multiplier for $\mathcal{D}$.*
- *$\mathcal{M} = \phi\mathcal{D}_\mu$ where $d\mu := |\phi^*|^2\, d\theta$.*

Invariant subspaces of $(M_z, \mathcal{D})$

Let $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D})$.

- Clearly $(M_z, \mathcal{M})$ satisfies properties (1),(2).
- If $\mathcal{M} \neq \{0\}$, then (3) also holds (Richter–Shields 1988).

Leads to:

Theorem (Richter 1991, Richter–Sundberg 1992)

Let $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D})$ and let $\phi \in \mathcal{M} \ominus M_z(\mathcal{M})$ with $\phi \not\equiv 0$. Then:

- *ϕ is a multiplier for $\mathcal{D}$.*
- *$\mathcal{M} = \phi \mathcal{D}_\mu$ where $d\mu := |\phi^*|^2 \, d\theta$.*

Corollary

$\mathcal{M}$ is cyclic (i.e. singly generated as an invariant subspace).

Cyclic invariant subspaces

Problem: Given $f \in \mathcal{D}$, identify $[f]_{\mathcal{D}}$, the closed invariant subspace of $\mathcal{D}$ generated by f.

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Theorem (Richter–Sundberg 1992)

Let $f \in \mathcal{D}$ have inner-outer factorization $f = f_i f_o$. Then

$$[f]_{\mathcal{D}} = f_i[f_o]_{\mathcal{D}} \cap \mathcal{D} = [f_o]_{\mathcal{D}} \cap f_i H^2.$$

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It remains to identify $[f_o]_{\mathcal{D}}$. We might expect that $[f_o]_{\mathcal{D}} = \mathcal{D}$. However, another phenomenon intervenes, that of boundary zeros.

Cyclic invariant subspaces and boundary zeros

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Let $f \in \mathcal{D}$ and let $E := \{f^* = 0\}$. Then $[f]_{\mathcal{D}} \subset \mathcal{D}_E$.

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Let $f \in \mathcal{D}$ and let $E := \{f^* = 0\}$. Then $[f]_{\mathcal{D}} \subset \mathcal{D}_E$.

Open problem

Let $f \in \mathcal{D}$ be outer and let $E := \{f^* = 0\}$. Then do we have $[f]_{\mathcal{D}} = \mathcal{D}_E$?
In particular, if $c(E) = 0$, then do we have $[f]_{\mathcal{D}} = \mathcal{D}$?

Special case where $c(E) = 0$ is a celebrated conjecture of Brown–Shields

Brown–Shields conjecture

$f \in \mathcal{D}$ is *cyclic* for $\mathcal{D}$ if $[f]_{\mathcal{D}} = \mathcal{D}$. Necessary conditions for cyclicity:

- f is outer;
- $E := \{f^* = 0\}$ is of capacity zero.

Conjecture (Brown–Shields, 1984)

These conditions are also sufficient.

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Partial results:

Theorem (Hedenmalm–Shields, 1990)

If $f \in \mathcal{D} \cap A(\mathbb{D})$ is outer and if $E := \{f = 0\}$ is countable, then f is cyclic.

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Theorem (El-Fallah–Kellay–Ransford, 2009)

If $f \in \mathcal{D} \cap A(\mathbb{D})$ is outer and if $E := \{f = 0\}$ satisfies, for some $\epsilon > 0$,

$$|E_t| = O(t^\epsilon) \ (t \to 0^+) \quad \textit{and} \quad \int_0^1 dt/|E_t| = \infty,$$

then f is cyclic.

Some further developments

- Shift-invariant subspaces and cyclicity in $\mathcal{D}_\mu$
  Richter, Sundberg (1992)
  Guillot (2012)
  El-Fallah, Elmadani, Kellay (2016)
- Optimal polynomial approximants
  Catherine Bénéteau and co-authors (2015 onwards)
- Cyclicity in Dirichlet spaces on the bi-disk
  Knese–Kosiński–Ransford–Sola (2019)

Conclusion: a shameless advertisement