Polynomial time guarantees for the Burer-Monteiro method

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Joint work with **Ankur Moitra** (MIT)
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and Algorithms for Geometric Constraint Systems
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Semidefinite programming

A *semidefinite program* is

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\text{(SDP)} \qquad \begin{aligned} \min_{X \in \mathbb{S}^n} \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \text{ for } i \in [m] \\ & X \succeq 0 \end{aligned}
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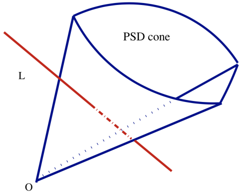

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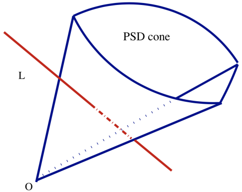

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- Many different applications.

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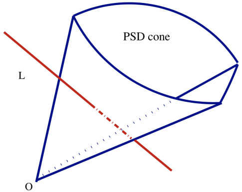

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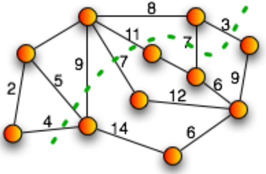
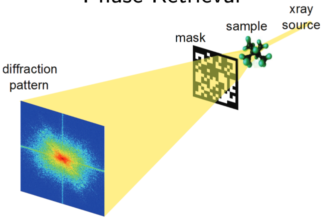
Goal: Show that SDPs can be solved in polynomial time with a more recent class of methods. The proof is remarkably geometric.

SDP applications

Max-Cut



Phase Retrieval



Matrix completion

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
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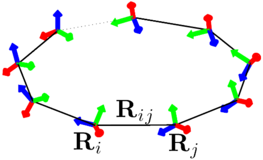
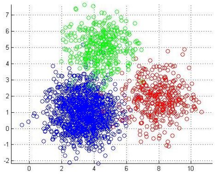
Rotation synchronization



Clustering



Solving large scale SDPs

Interior point methods intractable for large SDPs (too much memory)

Practical methods

- Low rank factorization (Burer, Monteiro)
- Frank-Wolfe / CGM (Hazan, Jaggi, Freund, Grigas, Mazumder)
- Sketching (Yurtsever, Ding, Udell, Tropp, Cevher)
- Bundle (Helmberg, Rendl, Oustry)
- Subgradients (Nesterov, Yurtsever, Tran Dinh, Cevher)

The Burer-Monteiro method is one of the most widely used in practice.

Burer-Monteiro method

(SDP) $$\min_{X \in \mathbb{S}^n} \quad C \bullet X \quad \text{s.t.} \quad A_i \bullet X = b_i \text{ for } i \in [m], \quad X \succeq 0$$

Assume the optimal solution has rank $\leq p$, and factorize

$$X = YY^T \quad \text{where } Y \in \mathbb{R}^{n \times p}$$

Use local optimization to solve the *nonconvex* problem

(BM) $$\min_{Y \in \mathbb{S}^n} \quad C \bullet YY^T \quad \text{s.t.} \quad A_i \bullet YY^T = b_i \text{ for } i \in [m]$$

How large should we choose p?

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Theorem (Barvinok-Pataki bound)

BM is equivalent to SDP when $p \gtrsim \sqrt{2m}$.

Known global guarantees on BM

Below Barvinok-Pataki bound

There might be spurious local minima [Waldspurger, Waters].

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Assume compactness and $p \gtrsim \sqrt{9m \log(\sigma^{-1})}$ for some $\sigma > 0$.

- Randomly perturb C (magnitude σ). No spurious approximate local minima w.h.p. [Pumir-Jelassi-Boumal]

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No polynomial time guarantees known!

Obstacle: How to find a local minimum that is exactly feasible?

Our results

Assumptions:

- $p \gtrsim \sqrt{(2+\eta)m}$ for a fixed $\eta > 0$.
- Constraint set is smooth and compact.
- Solve BM using local method with 2nd order guarantees.

Theorem (Polytime optimality)

Randomly perturb C (magnitude σ). Given an initial approx feasible point, then BM computes a point that is approx feasible & approx optimal w.h.p. in $\mathrm{poly}(n, \sigma^{-1})$ *iterations.*

Smoothed analysis [Spielman-Teng '01]

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BM for SDP feasibility

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Consider the least squares problem:

$$(\text{SDP}_{ls}) \qquad \min_{X \in \mathbb{S}^n} \sum_i (A_i \bullet X - b_i)^2 \quad \text{s.t.} \quad X \succeq 0$$

The associated Burer-Monteiro problem is

$$(\text{BM}_{ls}) \qquad \min_{Y \in \mathbb{S}^n} \sum_i (A_i \bullet YY^T - b_i)^2$$

Previous work on BM_{ls} relies on RIP [Bhojanapalli-Neyshabur-Srebro].

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Randomly perturb $A_1, \ldots, A_m$ (magnitude σ). Then BM_{ls} computes an approx feasible point w.h.p. in $\text{poly}(n, \sigma^{-1})$ *iterations.*

Spurious critical points

A critical point Y of BM is spurious if YY^T is not optimal for SDP.

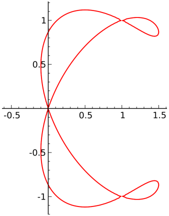
Theorem (Boumal-Voroninski-Bandeira)

Spurious critical points may only exist if

$$C \in \mathcal{M} + \mathcal{L} \subset \mathbb{S}^n$$

where $\mathcal{M} = \{S : \text{rank} S \leq n - p\}$ *and* $\mathcal{L} = \text{span}\{A_1, \ldots, A_m\}$

- Assume that $p \gtrsim \sqrt{2m}$.
- $\mathcal{M} + \mathcal{L}$ is an *algebraic variety* of codimension $\approx (p^2/2 - m)$.
- Generically, no spurious points.



Spurious approximate critical points

An ϵ-critical point Y of BM is spurious if YY^T is not δ-optimal for SDP, with $\delta = O(\epsilon)$.

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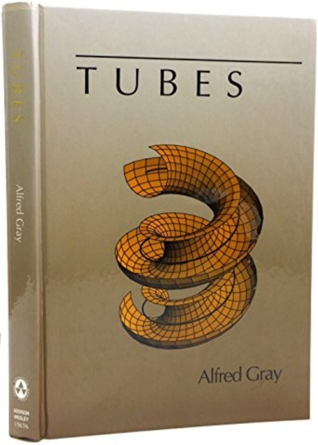
Theorem (C.-Moitra)

Spurious ϵ-critical points may only exist if

$$C \in \text{tube}_\epsilon(\mathcal{M} + \mathcal{L}) \subset \mathbb{S}^n$$

where $\text{tube}_\epsilon(\mathcal{W}) := \{X : \text{dist}(X, \mathcal{W}) \leq \epsilon\}$



Volumes of tubes

Theorem (Weyl 1939)

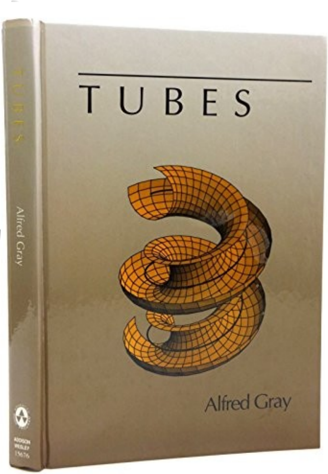
Let $V \subset \mathbb{R}^n$ manifold of codimension c. There are curvature constants $k_i(V)$ such that

$$\mathrm{Vol}[\mathrm{tube}_\epsilon(V)] = \sum_{i=c}^{n} k_i(V)\,\epsilon^i$$

Theorem (Lotz 2015, Basu-Lerario 2021)

Let $V \subset \mathbb{R}^n$ variety of codimension c defined by polynomials of degree D. Let x uniformly distributed on a ball of radius σ. Then

$$\Pr[x \in \mathrm{tube}_\epsilon(V)] \le O(nD\epsilon/\sigma)^c$$



Polytime optimality

Theorem (C.-Moitra)

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Proof.

Method converges to an ϵ-critical point in $\mathrm{poly}(\epsilon^{-1})$. Using tubes,

$$\Pr[\text{spurious}] \;\leq\; \Pr[C \in \mathrm{tube}_\epsilon(\mathcal{M}+\mathcal{L})] \;\leq\; \epsilon^{p^2/2-m} \cdot O\left(n^3/\sigma\right)^{p^2/2}$$

For $p > \sqrt{(2+\eta)m}$ and $\epsilon = O(\sigma/n^3)^{1+1/\eta}$ the probability is tiny. □

Polytime feasibility

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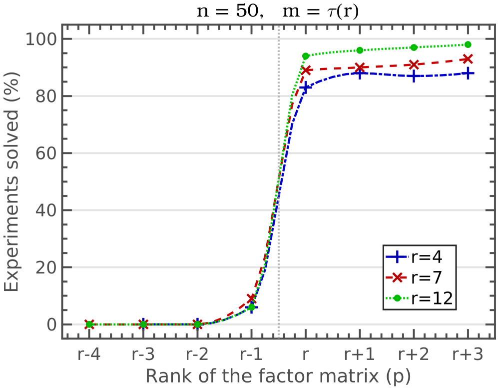
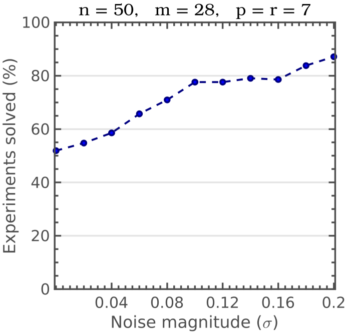
Experiments

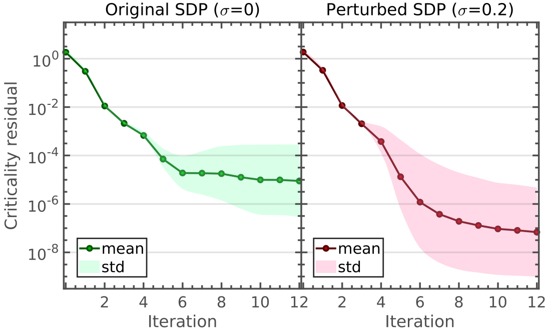
- Generate planted matrix of rank r.
- Generate SDP for which planted marix is optimal.
- Solve BM using augmented Lagrangians.



Experiments

- Fix an SDP instance for which BM behaves badly.
- Perturb the problem with small noise.
- Solve BM using augmented Lagrangians.





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- Guarantees work arbitrarily close to the Barvinok-Pataki bound.
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- D. Cifuentes, A. Moitra, *Polynomial time guarantees for the Burer-Monteiro method*, `arXiv:1912.01745`.
- D. Cifuentes *On the Burer-Monteiro method for general semidefinite programs*, Optimization Letters (2021): 1-11.

Thanks for your attention