# Rigidity with few locations

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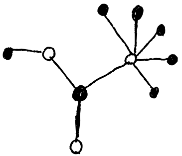
# Warmup:

Can you rigidify $G$ in $\mathbb{R}^d$ using $d+1=\chi(G)$ locations?

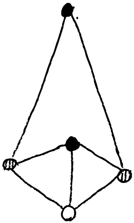
$(G,\rho)$

$\rho: V(G) \longrightarrow A$



$\chi=2$

$\mathbb{R}^1 \supseteq A$



$\chi=3$

$\mathbb{R}^2 \supseteq A$

# Main objective

$G = (V, E)$ is a finite graph.
$A \subseteq \mathbb{R}^d$.

**Definition**

$G$ is *A-rigid* if there exists $p : V \to A$ such that the framework $(G, p)$ is infinitesimally rigid.

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$\mathcal{F}$ a family of generically $d$-rigid graphs. $\mathcal{F}(n)$ is its subfamily of $G \in \mathcal{F}$, $|V(G)| \leq n$.

**Definition**

$\mathcal{F}$ is *d-rigid with c locations* if there exists $A \subseteq \mathbb{R}^d$ of size $c$ s.t all $G \in \mathcal{F}$ are $A$-rigid.
$c_d(\mathcal{F})$ is the minimal such $c$.

Main interest:

1. Families with bounded $c_d(\mathcal{F})$.
2. Growth of $c_d(\mathcal{F}(n))$ as $n \to \infty$.

# All $d$-rigid graphs

$\mathcal{F}_d = \{$all $d$-rigid graphs$\}$.

**Fekete-Jordan 2005**

$c_1(\mathcal{F}_1) = 2$. (As a spanning tree is bipartite.)
$c_2(\mathcal{F}_2(n)) = \Omega(\sqrt{n})$.

Their argument shows: for $d \geq 2$, $c_d(\mathcal{F}_d(n)) = \Omega(n^{1/d})$.
Sketch: let $H$ be minimally $d$-rigid on $k \geq d$ vertices. $G = G(H)$ obtained by connecting a new vertex to each subset of $d$ vertices of $H$. Then $|V(G)| = k + \binom{k}{d}$, and in a $d$-rigid realization of $G$ each vertex of $H$ must have a different location.

**Király 2021**

$c_2(\mathcal{F}_2(n)) = \Theta(\sqrt{n})$.

# Planar graphs

**Király 2021 (answers Whiteley)**

$c_2(\text{Planar Laman}) \leq 26.$

**Adiprasito-N. 2020**

$c_3(\text{Maximal Planar}) \leq 76.$

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The main algebraic statement, allowing inductive proofs in both results, is about moving vertices into $A$:

**Adiprasito-N. 2020, also Király 2021**

Assume $(G, p)$ is infinitesimally rigid in $\mathbb{R}^d$, $\deg_G(v) = c$, $A \subseteq \mathbb{R}^d$ with generic coordinates, $|A| \geq \binom{d+c}{d}$.
Then there exists $a \in A$ s.t. $(G, p')$ is infinitesimally rigid, where $p' : V \to \mathbb{R}^d$ is defined by $p'(v) = a$ and $p'(u) = p(u)$ for all $u \in V - v$.

# Graphs on surfaces

Let $M_g$ denotes the surface of genus $g$ (orientable or not).
Let $\mathcal{F}(M_g)$ be the family of graphs of triangulations of $M_g$.

**Fogelsanger 1988**

$c_3(\mathcal{F}(M_g)) \leq \aleph_0$ for all $g$, namely, every triangulated surface has a 3-rigid graph.

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Let $\mathcal{L}(M_g)$ be the Laman graphs embedable in $M_g$.

**Király 2021**

$c_2(\mathcal{L}(M_g)) \leq c(g)$ for some constant $c(g) = O(\sqrt{g})$.

Open: are there absolute constants $c_3, c_2$ s.t. for all $g$:
$c_3(\mathcal{F}(M_g)) \leq c_3$?
$c_2(\mathcal{L}(M_g)) \leq c_2$?

# New results

**Intermediate problem**

Does every graph of a triangulation of a surface $M_g$ contain a vertex spanning planar Laman subgraph?

Note: if YES then $c_2(\mathcal{F}(M_g)) \leq 26$.

**Theorem (N.- Simion Tarabykin)**

YES if the Euler characteristic $\chi(M_g) \geq 0$.

# More strongly

## N.-Tarabykin

- Every triangulation of the projective plane $\mathbb{R}P^2$ contains a spanning disc.
- Every triangulation of the Torus $T$ contains a spanning cylinder.
- Every triangulation of the Klein bottle $K$ contains a vertex spanning, planar, strongly connected 2-complex; it is either a cylinder, or a pinched disc, or a connected sum of two triangulated discs along a triangle.

Then these vertex spanning subcomplexes are indeed 2-rigid, hence contain a spanning planar Laman subgraph, and Király's result apply.

## Irreducible triangulations

A triangulation $\Delta$ of $M_g$ is irreducible if each contraction of an edge of $\Delta$ changes the tolopogy; equivalently, each edge belongs to an *empty* triangle of $\Delta$.

**Barnette-Edelson 1988/9**

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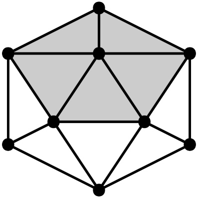
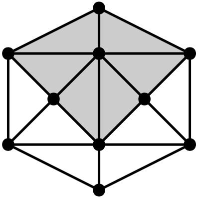
When $\chi(M_g) \geq 0$ the minimal triangulations are characterized:
2 such $\mathbb{R}P^2$: Barnette 1982.
21 such $T$: Lavrenchenko 1990.
29 such $K$: Lavrenchenko-Negami 1997, Sulanke 2006.

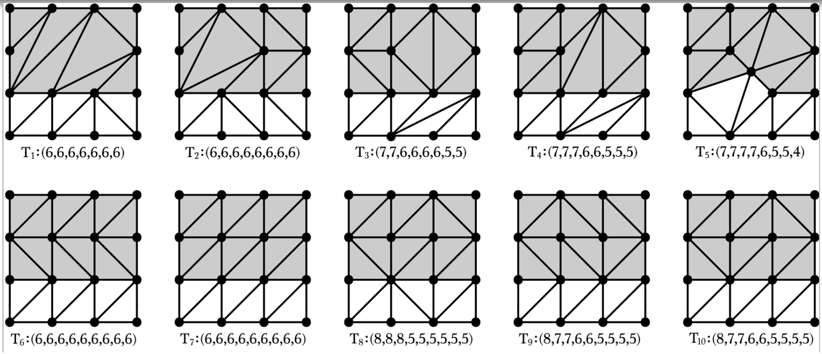


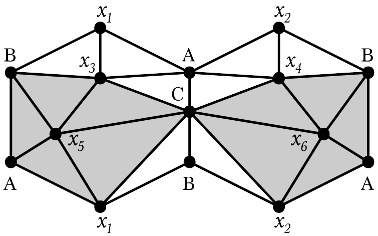
$RP2_1$:(5,5,5,5,5,5)



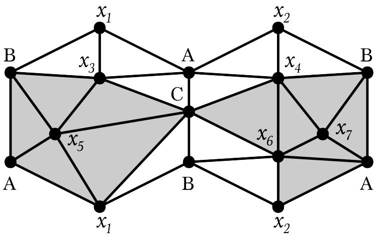
$RP2_2$:(6,6,6,6,4,4,4)

# Some spanning subcomplexes in minimal triangulations




$Kc_1$:(8,8,8,5,5,5,5,5,5)




$Kc_2$:(9,9,7,6,6,5,5,5,4,4)

# Induction via vertex splits

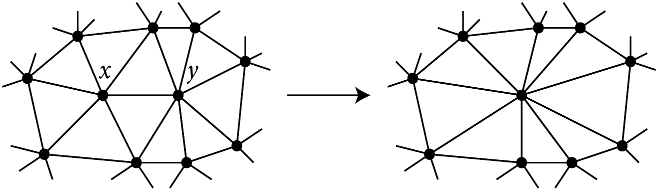
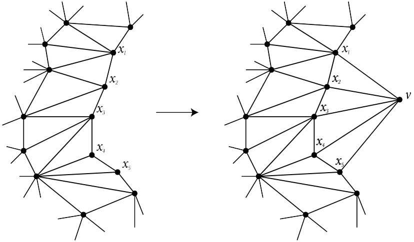



Figure: Above: edge contraction; reverse arrow for vertex split. Below: cone over boundary interval in the spanning subsurface.




We want the vertex split $\Delta \to \Delta'$ to allow an extension $S' \subseteq \Delta'$ of the spanning disc/cylinder/etc $S \subseteq \Delta$.

# Extension

**Definition: extension**

A spanning subsurface $S \subseteq \Delta$ is *extendible* if for every vertex split $\Delta \to \Delta'$ there exists $S' \subseteq \Delta'$ s.t. either
(i) $S'$ is obtained from $S$ by a split at the same vertex, or
(ii) $S'$ is obtained from $S$ by coning over an interval in its boundary.

Note: then $S' \subseteq \Delta'$ is spanning and homeomorphic to $S$.

**Theorem (N.-Tarabykin)**

Let $\Delta$ triangulate some $M_g$, and let $S \subseteq \Delta$ be a vertex spanning subsurface. Then:
(1) $S$ is extendible in $\Delta$ iff it includes at least one edge from every triangle in $\Delta$.
(2) If $S$ is extendible then it has an extendible extension $S' \subseteq \Delta'$.

# The easy direction

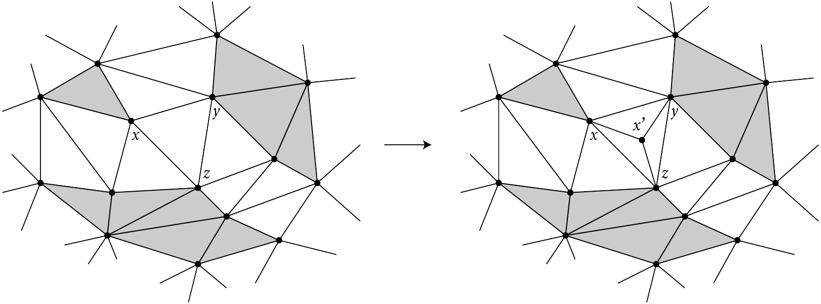



Figure: Non extendible subcomplex.

Note: all $S$ we chose in irreducible triangulations are extendible subsurfaces, except for the 4 in the "crosscap" triangulations of the Klein bottle, which are *pinched* discs.

# How to choose an extendible $S'$?

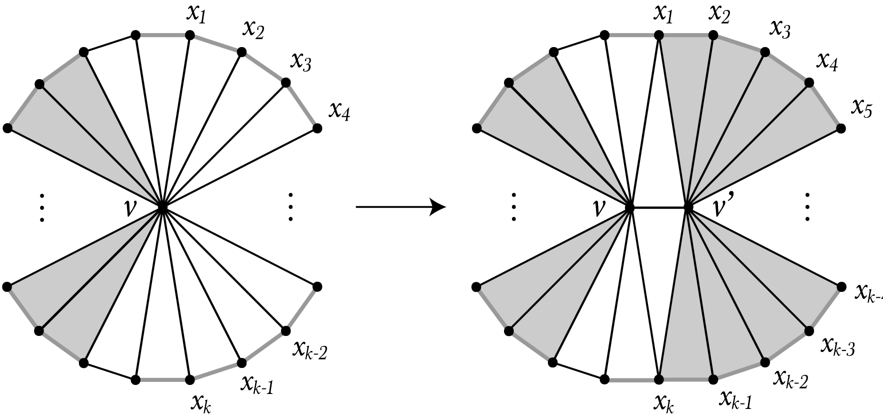



Figure: $S \to S'$ via coning over boundary interval.

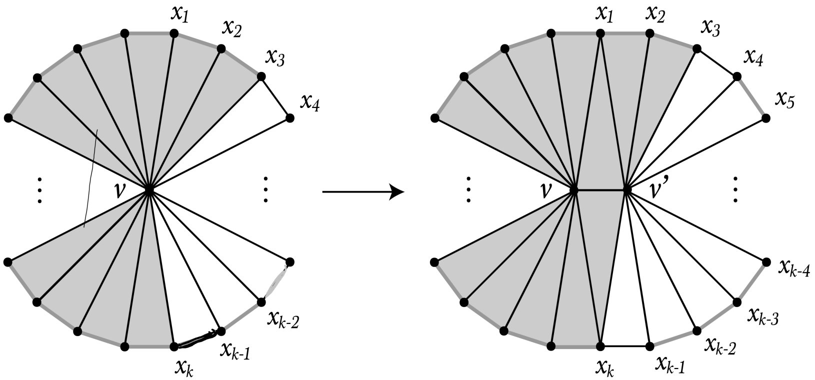
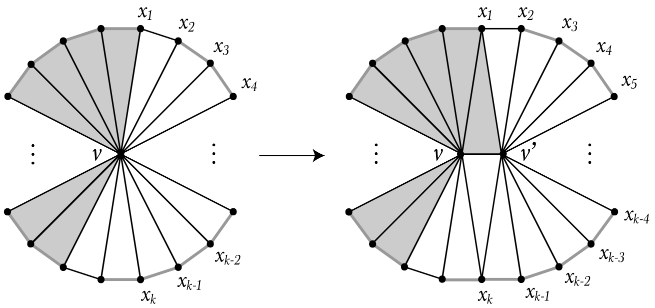
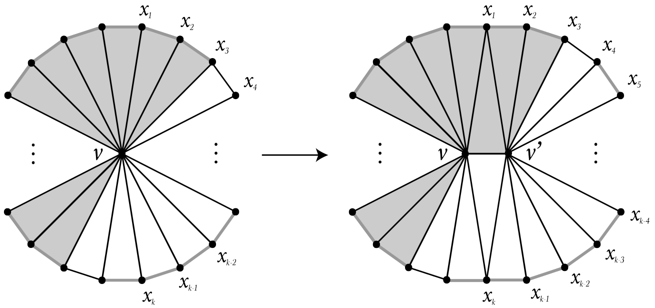



Figure: $S \to S'$ via vertex split.



$x_1$
$x_2$
$x_3$
$x_4$
$v$
$x_{k-2}$
$x_{k-1}$
$x_k$
$x_1$
$x_2$
$x_3$
$x_4$
$x_5$
$v$
$v'$
$x_{k-4}$
$x_{k-3}$
$x_{k-2}$
$x_{k-1}$
$x_k$

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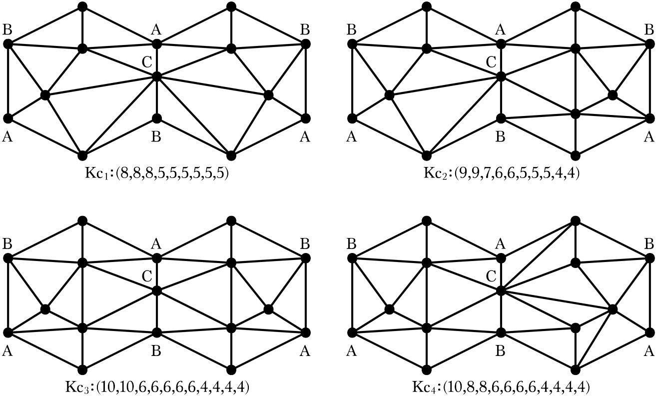
$x_{k-3}$
$x_{k-2}$
$x_{k-1}$
$x_k$

# What's left: crosscap triangulation of $K$

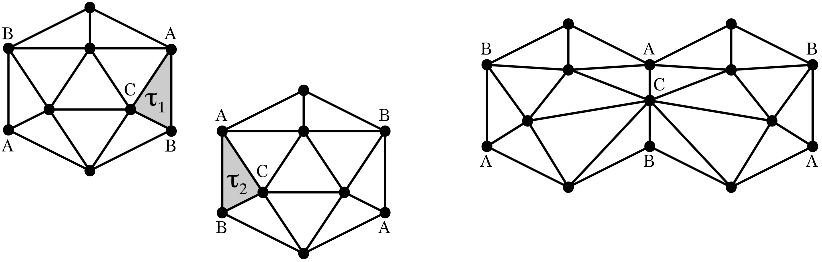
$ABC$ is a noncontractible cycle in each of the 4 crosscap irreducible triangulations:




Each is a connected sum of two $\mathbb{R}P^2$'s along the triangle $ABC$; this triangle can be part of the spanning disc in each $\mathbb{R}P^2$.

# Does $ABC$ survive the vertex splits?

If YES, then again the triangulation $\Delta$ of $K$ is a connected sum of two $\mathbb{R}P^2$'s along $ABC$, and $ABC$ can be taken as a triangle in each spanning disc.
The connected sum of those discs is a planar strongly connected 2-complex.



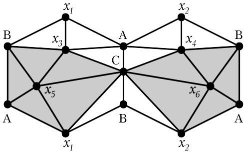
# If NOT

Then some vertex split induced a vertex split of the cycle $ABC$.

**Commutativity claim**

The vertex splits can be rearranged s.t. the first one splits $ABC$.

If $C$ splits first (similaly for $A, B$), choose $S$ a spanning pinched

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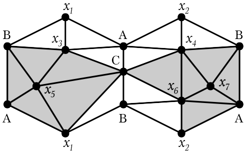


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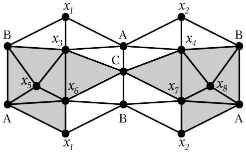
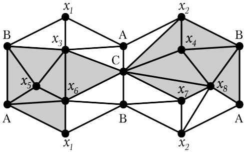



$Kc_3$:(10,10,6,6,6,6,6,6,4,4,4,4)




$Kc_4$:(10,8,8,6,6,6,6,6,4,4,4,4)

disc at $C$.

# From a pinched disc to a cylinder

The first split makes $S'$ a spanning cylinder.

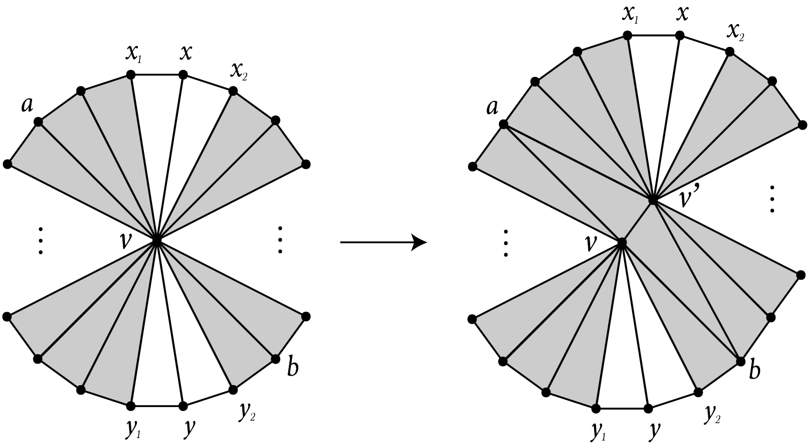



Figure: Resolving a singularity.

The Extension Theorem shows that further splits preserve having a spanning cylinder. This completes the proof. □

# Open problems

- All surfaces: (i) is $c_3(\mathcal{F}(M_g))$ uniformly bounded, namely by a constant independent of $g$? (ii) same for $c_2(\mathcal{L}(M_g))$? (iii) if NO in (i), then same for $c_2(\mathcal{F}(M_g))$? E.g. via:

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THANK YOU!