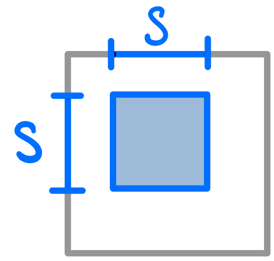
# DETERMINANTAL POLYNOMIALS & THE PRINCIPAL MINOR MAP

joint work with Abeer Al Ahmadieh (UW) arXiv:2105.13444

Principal Minor Map: $\varphi : \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{2^n}$

$A \mapsto (A_S)_{S\subseteq[n]}$




$A_S = \det(\square)$

$A_\emptyset = 1$

Goal: Cut out image of $\varphi$ with polynomial equations and inequalities.

Ex: $(n=2)$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ $\varphi(A) = (A_\emptyset, A_1, A_2, A_{12})$

$= (1, a_{11}, a_{22}, a_{11}a_{22} - a_{12}^2)$

$\text{image}(\varphi) = \{(1, A_1, A_2, A_{12}) : A_1A_2 - A_{12} \geq 0\}$

$a_{11} = A_1 \quad a_{22} = A_2 \quad a_{12} = \pm\sqrt{A_1A_2 - A_{12}}$

Properties of image of $\varphi$ Holtz, Sturmfels 2006

- closed, semialgebraic subset of $\mathbb{R}^{2^n}$]
- dimension $= \dim(\mathbb{R}^{n\times n}_{sym}) = \binom{n+1}{2}$
- invariant under action of $S_n$
- invariant under action of $SL_2(\mathbb{R})^n$

Ex $(n=3)$ $\dim = \binom{4}{2} = 6$ in $\{A_\emptyset = 1\} \subseteq \mathbb{R}^8$

Thm (HS '06) $(A_S)$ = princ. minors of $A \in \mathbb{R}^{3\times 3}_{sym} \Rightarrow$

$$A_\emptyset^2A_{123}^2 + A_1^2A_{23}^2 + A_2^2A_{13}^2 + A_3^2A_{12}^2 + 4\cdot A_\emptyset A_{12}A_{13}A_{23} + 4\cdot A_1A_2A_3A_{123}$$
$$-2\cdot A_\emptyset A_1A_{23}A_{123} - 2\cdot A_\emptyset A_2A_{13}A_{123} - 2\cdot A_\emptyset A_3A_{12}A_{123} - 2\cdot A_1A_2A_{13}A_{23}$$
$$-2\cdot A_1A_3A_{12}A_{23} - 2\cdot A_2A_3A_{12}A_{13} = 0.$$

$\parallel$

HYPDET$(A_S)$ "Cayley's Hyperdeterminant" of a $2\times2\times2$ tensor

For $n \geq 3$, HYPDET and all images under $SL_2(\mathbb{R})^n \rtimes S_n$ vanish

Thm (Oeding 2011)

$(A_S)_{S \subseteq [n]} = \varphi(A)$ $\iff$ $\mathrm{HYPDET}(\gamma \cdot (A_S)) = 0$

for some $A \in \mathbb{C}^{n\times n}_{sym}$ for all $\gamma \in SL_2(\mathbb{R})^n \rtimes S_n$.

$\mathbb{R}$

$+ A_i A_j - A_{ij} \geq 0$

---

This talk: Understand ↑ through determinantal representations of multiaffine polynomials

$$A \in \mathbb{R}^{n\times n}_{sym} \to \quad f = \det\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} + A\right) = \sum_{S \subseteq [n]} A_S \prod_{i \notin S} x_i$$

Ex $(n=2)$

$$f = \det\begin{pmatrix} x_1 + a_{11} & a_{12} \\ a_{12} & x_2 + a_{22} \end{pmatrix} = x_1 x_2 + a_{11} x_2 + a_{22} x_1 + a_{11} a_{22} - a_{12}^2$$

Revised (equivalent) goal: Characterize determinantal polynomials in $\mathbb{R}[x_1, \dots, x_n]_{MA}$

Group actions on $\mathbb{R}[x_1, \dots, x_n]_{MA}$:

$\pi \in S_n$ $\qquad \pi \cdot f = f(x_{\pi(1)}, \dots, x_{\pi(n)})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f = (cx_1 + d) f\left(\frac{ax_1 + b}{cx_1 + d}, x_2, \dots, x_n\right)$

## RAYLEIGH DIFFERENCES

$i, j \in [n]$

$f \in \mathbb{R}[x_1, \dots, x_n]_{MA}$

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$\in \mathbb{R}[x_k : k \neq i, j]_{MQ}$ (deg $\leq 2$ in each var)

Ex: $f = x_1 x_2 + x_1 x_3 + x_2 x_3$

$$\Delta_{12}(f) = (x_2 + x_3)(x_1 + x_3) - f \cdot 1 = x_3^2$$

Thm (Brändén, 2007)

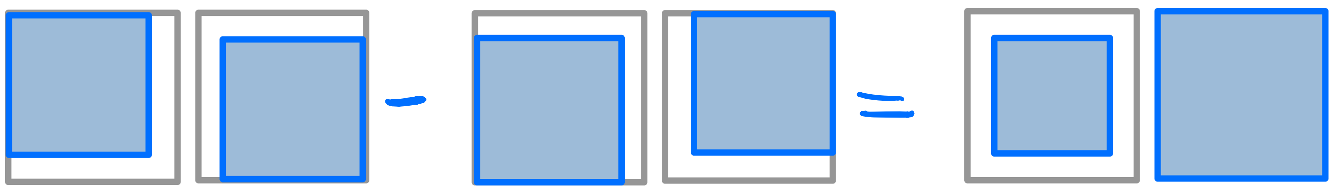
$f\in\mathbb{R}[x_1,\dots,x_n]_{MA}$ is stable $\iff$ $\Delta_{ij}(f)\geq 0$ on $\mathbb{R}^n$ for all $i,j\in[n]$

Thm (Kummer, Plaumann, V. 2013)

$f\in\mathbb{R}[x_1,\dots,x_n]_{MA}$ is determinantal $\iff$ $\Delta_{ij}(f)$ is a square in $\mathbb{R}[x_1,\dots,x_n]$ for all $i,j\in[n]$

IDEA for ($\Rightarrow$): DODGSON CONDENSATION (1860's)

$M \xrightarrow{\text{remove row/col } i} M_{ii}$



$$|M_{i,i}|\cdot|M_{j,j}| - |M_{i,j}|\cdot|M_{j,i}| = |M_{ij,ij}|\cdot|M|$$

For $M=\begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_n\end{pmatrix}+A$
$$\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j} - (\text{a square}) = \frac{\partial^2 f}{\partial x_i\,\partial x_j}\cdot f$$

Ex ($n=3$) $\quad f=\sum_{S\subseteq[3]} A_S \prod_{i\notin S} x_i \qquad \Delta_{12}(f)\in\mathbb{R}[x_3]_{\leq 2}$

$= a x_3^2 + b x_3 + c$

$\Delta_{12}(f)$ = a square $\iff$ $\underbrace{b^2-4ac}_{=\text{HYPDET}(A_S)}=0,\ a, c\geq 0$

$a \leftarrow A_1A_2 - A_{12}$

Thm (Al Ahmadieh, V, 2021)

$P\in\mathbb{R}[x_1,\dots,x_n]_{MQ}$ is a square

$\iff$ $\gamma\cdot P|_{x_2=\dots=x_n=0}$ is a square in $\mathbb{R}[x_1]$ for all $\gamma\in SL_2(\mathbb{R})^n\rtimes S_n$.

Cor: $(A_S)_{S\subseteq[n]}=\varphi(A)$ for some $A\in \mathbb{R}^{n\times n}_{sym}$ $\iff$ $\gamma\cdot\text{HYPDET}(A_S)=0$ and $\gamma\cdot(A_1A_2-A_{12}A_\emptyset)$ $\cancel{\geq 0}$ square in $R$ for all $\gamma\in SL_2(\mathbb{R})^n\rtimes S_n$.

$R$

$R$ = any UFD except $\mathbb{F}_3$

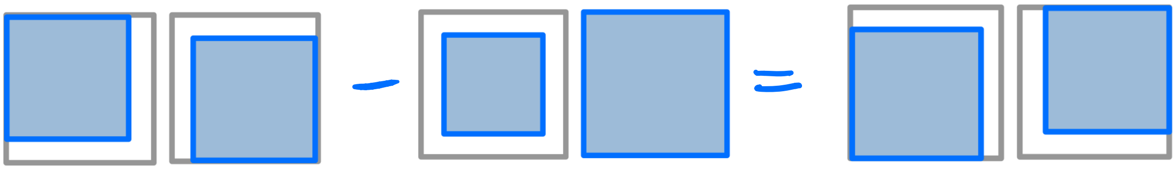
# Generalizing to Hermitian matrices

$\mathcal{H}_n = \{n \times n \text{ Hermitian matrices}\}$

Q: What is the image of $\mathcal{H}_n$ under $\varphi(A) = (A_S)_{S \subseteq [n]}$?

For $f = \det\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} + A\right)$, $A \in \mathcal{H}_n$,

Dodgson condensation $\Rightarrow$ $\Delta_{ij}(f) = g\bar{g}$ for $g = |M_{ij}| \in \mathbb{C}[x_1, \dots, x_n]$



Thm (Al Ahmadieh, V)

$f = \det\left(\underbrace{\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}} + A\right)$ for some $A \in \mathcal{H}_n$ $\iff$ for all $i,j$, $\Delta_{ij}(f)$ is a Hermitian square $g\bar{g}$

$(n=3)$ $\Delta_{12}(F) = a x_3^2 + b x_3 + c = g\bar{g} \iff \underbrace{b^2 - 4ac \leq 0}_{\text{HYDET}(A_S)}$, $a, c \geq 0$

Thm: A polynomial $P \in \mathbb{R}[x_1, \dots, x_n]_{MQ}$ is a Hermitian square

$\iff$ $\forall \gamma \in SL_2(\mathbb{R})^n \rtimes S_n$, $\gamma \cdot P|_{x_3 = \dots = x_n = 0}$ is a Herm. sq.

$\iff$ coeff. of $P$ satisfy some inequalities and equations of degree 6.

Cor: The image of Hermitian matrices under $\varphi(A) = (A_S)_{S \in [n]}$ is cut out by the orbit of

$$A_1 A_2 - A_{12} A_\emptyset \geq 0, \quad \text{HYPDET}(A) \leq 0, \quad \text{equations of deg } 12$$

under $SL_2(\mathbb{R})^n \rtimes S^n$.