Zero-sum cycles: a necessary condition for the flexibility of polyhedra

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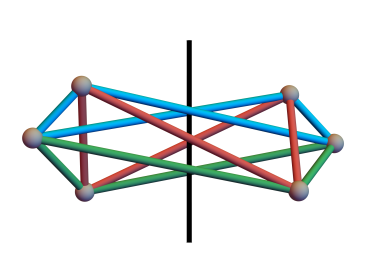
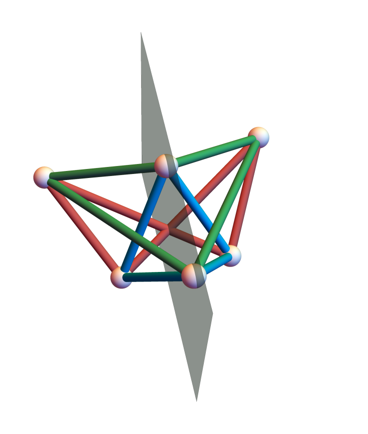
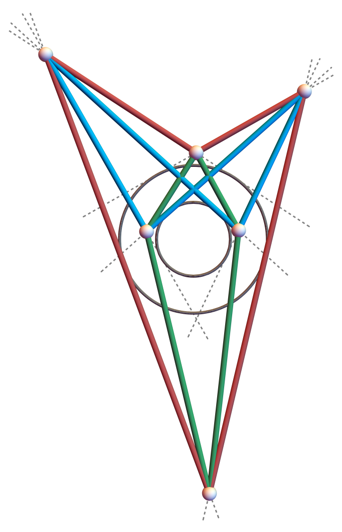
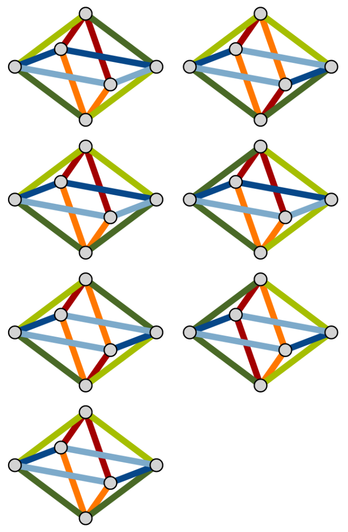
Bricard's octahedra I&II





Bricard's octahedra III





Definitions

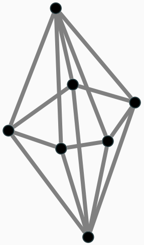
A *triangular polyhedron* is a finite two-dimensional abstract simplicial complex s. t. every edge belongs to exactly two faces.

A *realization* of a triangular polyhedron whose 1-skeleton is $G = (V, E)$ is a map $\rho \colon V \longrightarrow \mathbb{R}^3$ such that $\rho(u) \neq \rho(v)$ for every $\{u, v\} \in E$. The realization ρ induces *edge lengths* $\lambda = (\lambda_e)_{e \in E}$ where $\lambda_{\{u,v\}} := \|\rho(u) - \rho(v)\| \in \mathbb{R}_{>0}$ for $\{u, v\} \in E$.

A *flex* of (G, ρ) is a continuous map $f \colon [0, 1) \longrightarrow (\mathbb{R}^3)^V$ such that

- $f(0)$ is the given realization ρ;
- for any $t \in [0, 1)$, the realizations $f(t)$ and $f(0)$ induce the same edge lengths;
- for any two distinct $t_1, t_2 \in [0, 1)$, the realizations $f(t_1)$ and $f(t_2)$ are not congruent.

Suspensions



Theorem (Connelly, 1974, Mikhalev, 2001)

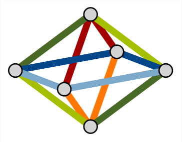
If a suspension has a flex, then there is a sign assignment such that the signed sum of lengths of the edges in the equator is zero.

Previous result

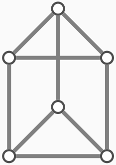
Theorem (Alexandrov, 2019)
Consider an oriented polyhedron with triangular faces that admits a flex. Let e be an edge of the polyhedron. If the dihedral angle at e changes along the flex, then the length of e is a $\mathbb{Q}$-linear combination of the lengths of the remaining edges.

Zero-sum cycle

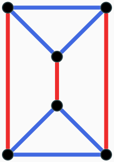
Theorem (Gallet, Grasegger, L., Schicho, 2020)
Consider a polyhedron with triangular faces that admits a flex. Let e be an edge of the polyhedron. If the dihedral angle at e changes along the flex, then there is an induced cycle of edges containing e and a sign assignment such that the signed sum of lengths of the edges in the cycle is zero.



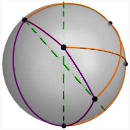
General approach



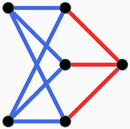
$\mathbb{P}^{|V|-2} \times \mathbb{P}^{|V|-2}$



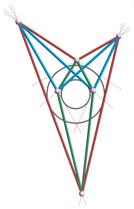
NAC-coloring



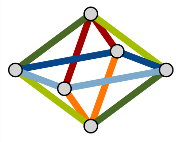
$\overline{\mathcal{M}}_{0,2|V|}$



NAP-coloring



$M^V \subset (\mathbb{P}^4)^V$



zero-sum cycle

Möbius embedding

We embed $\mathbb{R}^3$ into $\mathbb{P}^4$ by the map

$$(x, y, z) \mapsto (x : y : z : x^2 + y^2 + z^2 : 1)\,.$$

Every point $(x : y : z : r : h)$ in the image of $\mathbb{R}^3$ lies on the hypersurface $M = \{x^2 + y^2 + z^2 - rh = 0\} \subset \mathbb{P}^4$.

- distance: $-\frac{1}{2}\left\|u_1 - u_2\right\|^2 = \langle p_1, p_2\rangle_M$, where $u_i \mapsto p_i$
- infinite points

Main result

Theorem

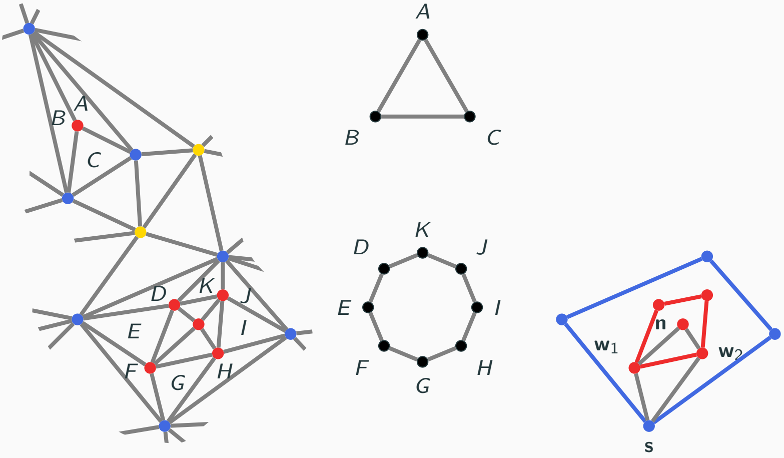
Consider a polyhedron with triangular faces that admits a flex. If the dihedral angle between faces $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{s}\}$ *and* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{n}\}$ *is not constant along the flex, then there is an induced cycle of edges containing* $\{\mathbf{w}_1, \mathbf{w}_2\}$ *but neither the vertex* $\mathbf{s}$ *nor* $\mathbf{n}$ *and there is a sign assignment such that the signed sum of lengths of the edges in the cycle is zero.*

Sketch of the proof

1. Fix the face $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{n}\}$.
2. Consider the Zariski closure of the image of the flex in M^V.
3. Pick an element ρ_∞ such that $\rho_\infty(\mathbf{s})$ is infinite.
4. Color each vertex v:

$$\begin{cases} \text{red} & \text{if } \rho_\infty(v) \text{ is finite,} \\ \text{blue} & \text{if } \rho_\infty(v) \text{ is simple infinite and } \Psi\big(\rho_\infty(v)\big) = \Psi\big(\rho_\infty(\mathbf{s})\big)\,, \\ & \text{where } \Psi(x:y:z:r:0) = (x:y:z)\,, \\ \text{gold} & \text{otherwise.} \end{cases}$$

Cycle construction



$$\mathrm{Fin}_{\rho_\infty(\mathbf{s})} := M \cap \mathbb{T}_{\rho_\infty(\mathbf{s})} M \cap \{h \neq 0\}$$

Sign assignment construction

Let $(v_1, \ldots, v_k, v_{k+1} = v_1)$ be the red cycle. There is a map $\pi : \mathrm{Fin}_{\rho_\infty(\mathsf{s})} \to \mathbb{C}$ such that

$$\lambda_{\{v_j, v_{j+1}\}} = \pm\Big(\pi\big(\rho_\infty(v_j)\big) - \pi\big(\rho_\infty(v_{j+1})\big)\Big).$$

Therefore we can choose integers $\eta_j \in \{1, -1\}$ such that

$$\sum_{j=1}^{k} \eta_j \, \lambda_{\{v_j, v_{j+1}\}}$$

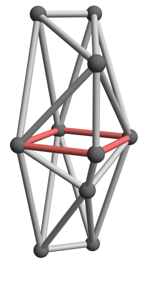
is a telescoping sum, which yields 0.

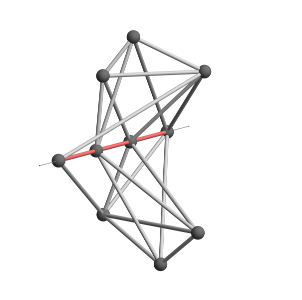
Generalizations

- The cycle can be chosen so that it consists only of edges whose dihedral angle changes along the flex.
- The theorem also holds for non-triangular polyhedra.

Butterfly motion

Let S be a cycle in the 1-skeleton of a polyhedron that separates the graph. For any sign assignment to the edges of S, not all having the same sign, the polyhedron admits a realizations with a flex such that the signed sum of the edge lengths induced by the realization in the cycle S is zero and the dihedral angles at all edges of S vary along the flex.





Thank you

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