A convex form that is not a sum of squares

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Forms: homogeneous polynomials of degree $2d$ in n variables

Convexity:

$$p(\tfrac{1}{2}(x+y)) \leq \tfrac{1}{2}(p(x)+p(y))$$

- All convex forms are nonnegative

Sums of squares:

$$p(x) = \sum_i [q_i(x)]^2$$

- Some nonnegative forms are SOS

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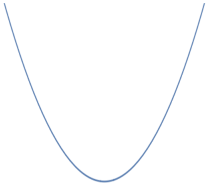
Are convex forms always sums of squares?

Blekherman (2012): NO! non-constructive argument

This talk: Explicit example of degree 4 in 272 variables

Context



Convexity:

- Natural assumption for opt.
- $\implies$ local optimization 'works'

Sums of squares:

- Natural sufficient condition for nonnegativity
- Can be verified by solving a semidefinite program

- Basic sos-based optimization approach

$$\max_{\gamma} \gamma \text{ s.t. } p(x) - \gamma \text{ SOS}$$

Are easy polynomial optimization problems (due to convexity) also easy for SOS?

When are convex forms sums of squares?

Positive results:

Classical:

Convex $\implies$ nonneg. $\implies$ SOS

- $n = 2$
- $2d = 2$
- $(n, 2d) = (3, 4)$ (Hilbert)

El Khadir (2020):

$(n, 2d) = (4, 4)$

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Negative results:

Blekherman (2012): For fixed degree $2d \geq 4$, and large enough n, there exist convex forms that are not SOS

If $2d = 4$ need $n \geq 27179089915$ to guarantee existence

How do we know a form is convex?

SOS-convexity: (Helton-Nie, Ahmadi-Parrilo, etc.)

$$\text{If} \quad \tfrac{1}{2}(p(x)+p(y)) - p(\tfrac{1}{2}(x+y)) \quad \text{is SOS}$$

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Blekherman's sufficient condition:

If p is a form of degree $2d$ and

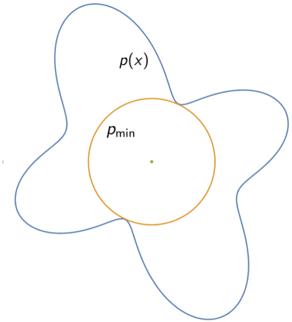
$$\left(1 - \frac{1}{2d-1}\right)\|x\|^{2d} \leq p(x) \leq \left(1 + \frac{1}{2d-1}\right)\|x\|^{2d}$$

then p is convex.

Polynomial optimization on the sphere

$$p_{\min} := \min_{\|x\|^2=1} p(x)$$

- If $2d = 2$, symmetric eigenvalue problem
- If $2d \geq 4$ NP-hard



Example (Motzkin-Straus): For graph $G = (V, E)$ define

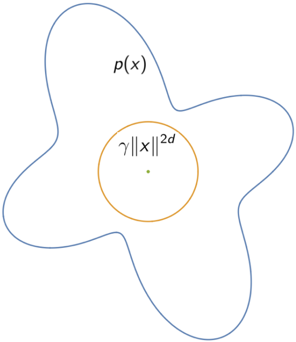
$$p_G(x) = \sum_{i \in V} x_i^4 + 2 \sum_{\{i,j\} \in E} x_i^2 x_j^2$$

Minimum over sphere is $1/\alpha(G)$

Basic SOS relaxation

$$p_{\min}^{\text{sos}} = \max_{\gamma} \gamma \ \text{ subject to } \ p(x) - \gamma \|x\|^{2d} \ \text{ SOS}$$

How good is the basic SOS relaxation?



Natural relative error measure:

$$\text{GAP}(p) = \frac{p_{\max} - p_{\min}^{\text{sos}}}{p_{\max} - p_{\min}} \geq 1.$$

Examples:

- $\text{GAP}(p_G) \leq 2$ for all graphs G
- $\text{GAP}(\text{Motzkin}) \approx 1.0046$

General upper bounds on $\text{GAP}(\cdot)$: Nie 2012, Fang-Fawzi 2020

Large GAP gives convex not-SOS forms

Recall: Looking for a form that is convex but not SOS

Intuition: Want p that is:

- near constant on sphere (controlled by $p_{\min}$ and $p_{\max}$)
- BUT hard to certify this using SOS (i.e., large $\text{GAP}(p)$)

Proposition: If p is a form of degree $2d$ and $\text{GAP}(p) > d$ then

$$p(x) - [dp_{\min} - (d-1)p_{\max}]\|x\|^{2d} \quad \text{is convex but not SOS}$$

Cauchy-Schwarz: the prototypical sum of squares

Degree 4 in $2k$ variables

$$\mathrm{CS}_{\mathbb{R}}(x, y) = \|x\|^2\|y\|^2 - \langle x, y\rangle^2 \geq 0.$$

On sphere in $\mathbb{R}^{2k}$: $\|x\|^2 + \|y\|^2 = 1$

- Minimum value $= 0$
- Maximum value $= 1/4$

Sum-of-squares: (follows from Cauchy-Binet over $\mathbb{R}$)

$$\mathrm{CS}_{\mathbb{R}}(x, y) = \det \begin{bmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{bmatrix} = \sum_{1 \leq i < j \leq k} (x_i y_j - y_i x_j)^2.$$

Cauchy-Schwarz: the prototypical sum of squares?

Over finite-dim. real normed division algebra ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$)

$$\mathrm{CS}_{\mathbb{O}}(x, y) = \|x\|^2\|y\|^2 - |(x, y)|^2$$

where $x_1, \ldots, x_k, y_1, \ldots, x_k \in \mathbb{O}$ (octonions) and

$$\|x\|^2 := \sum_i x_i^* x_i \quad \text{and} \quad (x, y) := \sum_i x_i^* y_i$$

On sphere in $\mathbb{R}^{16k}$: $\|x\|^2 + \|y\|^2 = 1$

- Minimum value $= 0$ (Cauchy-Schwarz inequality!)
- Maximum value $= 1/4$

Theorem: (Ge-Tang 2018) $\mathrm{CS}_{\mathbb{H}}$ and $\mathrm{CS}_{\mathbb{O}}$ are not SOS!

Main result

Recall: $\mathrm{CS}_{\mathbb{O}}(x, y)$ quartic form in $16k$ real variables

Theorem: (S. 2021) If $k \geq 17$ then $\mathrm{GAP}(\mathrm{CS}_{\mathbb{O}}) = \frac{15}{8/k+7} > 2$

Corollary: If $k = 17$ then the quartic

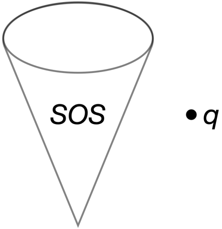
$$q(x, y) := \frac{1}{4}(\|x\|^2 + \|y\|^2)^2 + \mathrm{CS}_{\mathbb{O}}(x, y)$$

in $16 \times 17 = 272$ variables is convex but not SOS.

Note: small perturbation $\longrightarrow$ strictly convex but not SOS form

Remaining proof ideas

Key issue: showing q is not SOS



- "Just" need to solve semidefinite feasibility problem
- Naïve formulation: $\binom{273}{2} \times \binom{273}{2}$ matrices

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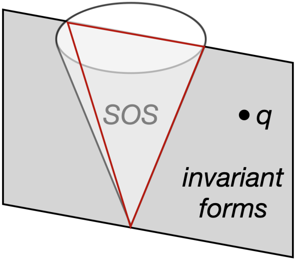

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- Symmetry: q invariant under
  $\mathrm{Spin}(9) \times O(k)$-action on $\mathbb{R}^{16\times k} \cong \mathbb{O}^{2\times k}$
- Symmetry reduction $\longrightarrow$ LP feasibility problem in $\mathbb{R}^3$

Questions: convex forms and sums of squares

- What is the real reason why this example works?
- How can we find examples of higher degree?
  Should this be "easier"?
- Can we construct examples in higher degree
  from lower degree examples?

Question: (Ahmadi-Parrilo) For which $(n, 2d)$ are n-variate convex forms of degree $2d$ always SOS?

Questions: optimization on the sphere

How big can $\text{GAP}(p)$ be for n-variate forms of degree $2d$?

El Khadir (2020): convex quaternary quartics are sos

Corollary: If p is a quaternary quartic form then $\text{GAP}(p) \leq 2$

Improves on bounds of Nie & Fang-Fawzi in this special case:
further improvements?

Problem: find explicit examples with large $\text{GAP}(\cdot)$

Thank You!

More information:

"A convex form that is not a sum of squares"

`https://arxiv.org/abs/2105.08432`