Numerical computation of monodromy action over $\mathbb{R}$

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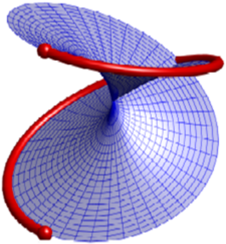
Duke
Trinity College of Arts & Sciences

Outline

Motivation

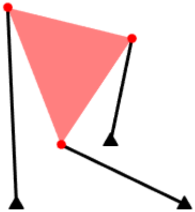
The complex monodromy group encodes information regarding the permutations of solutions to a polynomial system over loops in the parameter space. It gives structural information in the following ways:

- symmetry of solutions
- some restrictions to number of real solutions
- decomposition of varieties into irreducible components



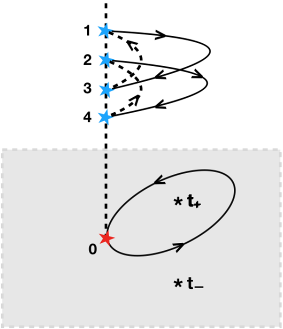
Motivation

Main question: How can we understand the behavior of real solutions over real loops in parameter space?



This idea influences many applications: in kinematics, it is related to nonsingular assembly mode change for parallel manipulators.

Complex monodromy group



- Fix a generic basepoint
- Assign an ordering of the solutions
- Pick a loop in the parameter space that avoids singularities
- How do the solutions permute along the loop?
- The collection of the permutations is the *complex monodromy group*.

Note: The complex monodromy group is independent of choice of basepoint and has an equivalent monodromy group when a general curve section of the parameter space is considered.

Example

Consider the parameterized polynomial system

$$F(x;p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

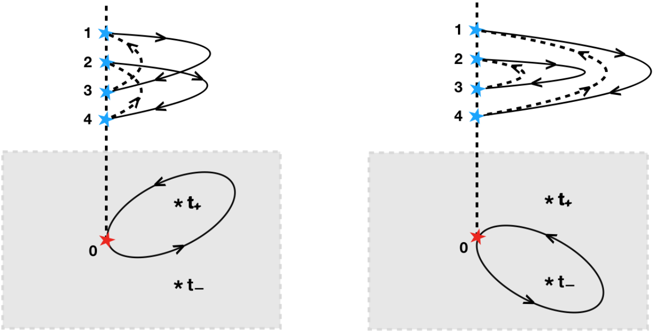
- Take basepoint $b = (1,0) \in \mathbb{C}^2$ such that $p_1^2 + p_2^2 \neq 0$
- Order the 4 nonsingular isolated solutions:

  $x^{(1)} = (1,0),\ \ x^{(2)} = (-1,0),\ \ x^{(3)} = (0,\sqrt{-1}),\ \ x^{(4)} = (0,-\sqrt{-1})$
- Restrict parameter space to the line parametrized by $\ell(t) = (1-t, 2t)$
  - This gives 2 singular points, $t_{\pm}$
- Loop around these singular points gives us two permutations:

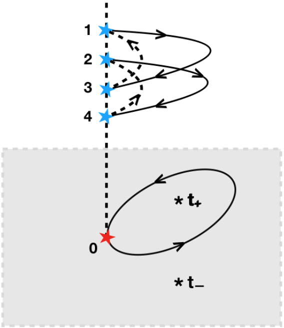
$$\sigma_{\gamma_+} = (1\ 3)(2\ 4) \qquad \text{and} \qquad \sigma_{\gamma_-} = (1\ 4)(2\ 3)$$

These generate the Klein group on four elements $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathcal{S}_4$

Example



Real monodromy group



- Fix a **real** basepoint
- Assign an ordering of the **real** solutions
- Pick a **real** loop in the **real** parameter space that avoids singularities
- How do the solutions permute along the loop?
- The collection of the permutations is the *real monodromy group*.

Note: This definition has restrictions: (1) only basepoint independent within the same connected component and (2) it's not clear how to slice.

Example 1

Consider the parameterized polynomial system

$$F(x;p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

- Take basepoint $b = (1,0) \in \mathbb{R}^2$ such that $p_1^2 + p_2^2 \neq 0$
- Order the 2 **real** nonsingular isolated solutions:

  $x^{(1)} = (1,0), \;\; x^{(2)} = (-1,0)$
- Loop around the singular point gives us the permutation:

  $\sigma_\gamma = (1\ 2)$
- Thus, the real monodromy group is $\mathcal{S}_2 = \{(1), (1\ 2)\}$.

Example 2

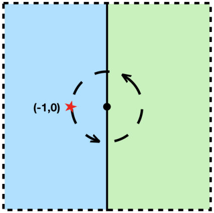
Consider a slightly modified parameterized polynomial system

$$F(x;p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

- Take basepoint $b = (-1, 0) \in \mathbb{R}^2$ such that $p_1^2 + p_2^2 \neq 0$
- Order the **real** 4 nonsingular isolated solutions:

  $x^{(1)} = (1,0),\ \ x^{(2)} = (-1,0),\ \ x^{(3)} = (0,1),\ \ x^{(4)} = (0,-1)$

- No nontrivial real loop exists around the singularity for all 4 solutions
- Fundamental group is trivial
- Thus, the real monodromy group is trivial



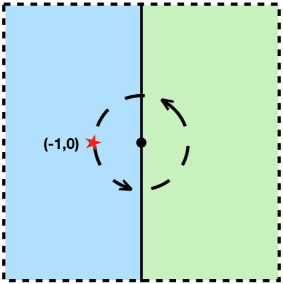
Real monodromy structure

$$F(x;p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0$$

Let's compute the real monodromy structure:

Consider $x^{(1)} = (1,0)$ along the loop shown.

Does it stay real and nonsingular along the loop?



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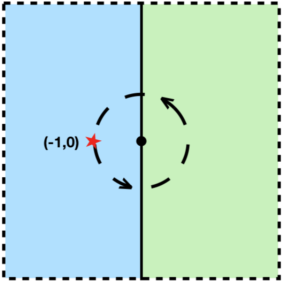
Does it stay real and nonsingular along the loop? **Yes**



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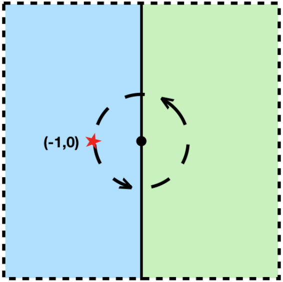
Does the solution permute?



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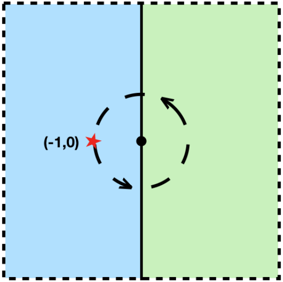

Does it stay real and nonsingular along the loop? **Yes**

Does the solution permute?

Yes, to $x^{(2)} = (-1,0)$

Real monodromy structure

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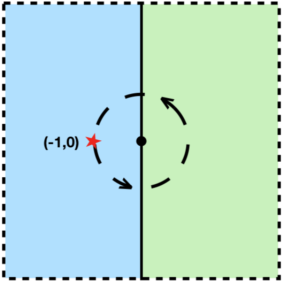
Does it stay real and nonsingular along the loop? **Yes**

Does the solution permute?

Yes, to $x^{(2)} = (-1,0)$

We represent this as:

- $\mathcal{G}_1$
  - $\{1\},\{2\} \mapsto \{\{1\},\{2\}\}$
  - $\{q_1\} \mapsto \{\{q_1\}\}$ for all others



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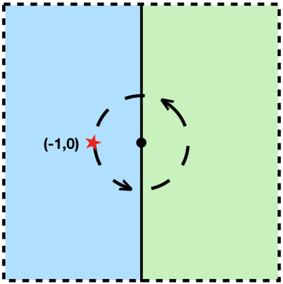
In general,we have:

$$\mathcal{G}_k : k\text{-ordered solutions} \to \text{sets of } k\text{-ordered solutions}$$

that can be attained by a real loop where all solutions in the set remain real and nonsingular.



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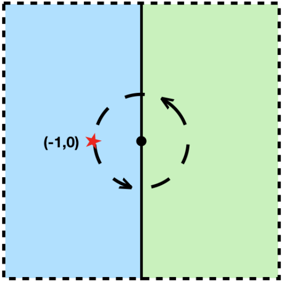
Next, consider the set of sols. $\{x^{(1)}, x^{(2)}\}$.

Do these **both** stay real and nonsingular along the loop?



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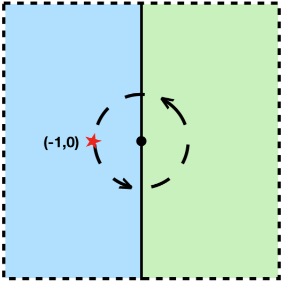

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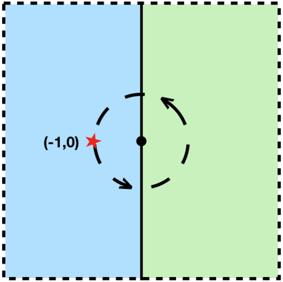
Do any permutations occur?



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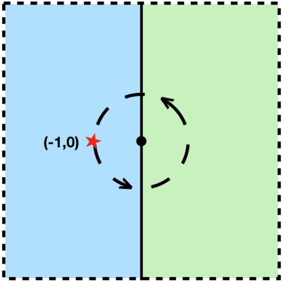
Do these **both** stay real and nonsingular along the loop? **Yes**

Do any permutations occur? **Yes**

$$\{x^{(1)}, x^{(2)}\} \rightarrow \{x^{(2)}, x^{(1)}\}$$



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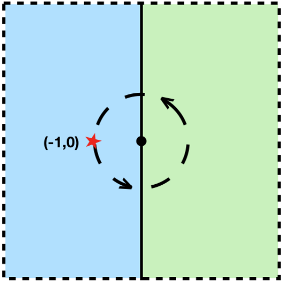


Continuing with all pairs, we obtain:

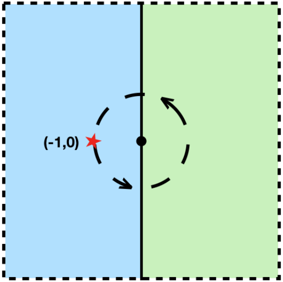
- $\mathcal{G}_2$
  - $\{1, 2\} \mapsto \{\{1, 2\}, \{2, 1\}\}$
  - $\{q_1, q_2\} \mapsto \{\{q_1, q_2\}\}$ for all others

Real monodromy structure

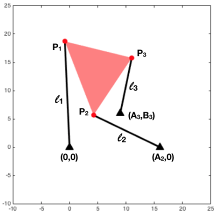
$$F(x;p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0$$

Continuing in this fashion, the real monodromy structure is:

- $\mathcal{G}_1$
  - $\{1\}, \{2\} \mapsto \{\{1\}, \{2\}\}$
  - $\{q_1\} \mapsto \{\{q_1\}\}$ for all others
- $\mathcal{G}_2$
  - $\{1, 2\} \mapsto \{\{1, 2\}, \{2, 1\}\}$
  - $\{q_1, q_2\} \mapsto \{\{q_1, q_2\}\}$ for all others
- $\mathcal{G}_3$
  - $\{q_1, q_2, q_3\} \mapsto \{\{q_1, q_2, q_3\}\}$
- $\mathcal{G}_4$
  - $\{q_1, q_2, q_3, q_4\} \mapsto \{\{q_1, q_2, q_3, q_4\}\}$



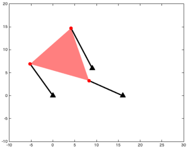
3RPR mechanism



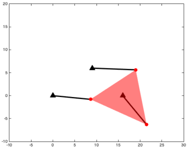
$$F(p,\phi;c)=\begin{bmatrix} \phi_1^2+\phi_2^2-1 \\ p_1^2+p_2^2-2(a_3p_1+b_3p_2)\phi_1+2(b_3p_1-a_3p_2)\phi_2 \\ \quad +a_3^2+b_3^2-c_1 \\ p_1^2+p_2^2-2A_2p_1+(a_2-a_3)^2+b_3^2+A_2^2-c_2 \\ \quad +2((a_2-a_3)p_1-b_3p_2+A_2a_3-A_2a_2)\phi_1 \\ \quad +2(b_3p_1+(a_2-a_3)p_2-A_2b_3)\phi_2 \\ p_1^2+p_2^2-2(A_3p_1+B_3p_2)+A_3^2+B_3^2-c_3 \end{bmatrix}$$

Fix $c_3 = 100$ and consider ℓ_1 and ℓ_2 as paramters. At the "home" position $c^* = (75, 70)$, the system $F(p,\phi;c^*) = 0$ has 6 nonsingular real solutions.

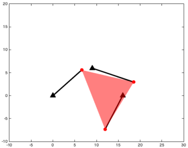
3RPR mechanism



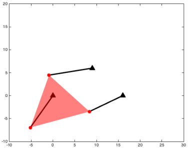
$x^{(1)}$



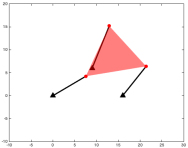
$x^{(2)}$



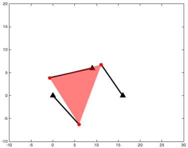
$x^{(3)}$



$x^{(4)}$



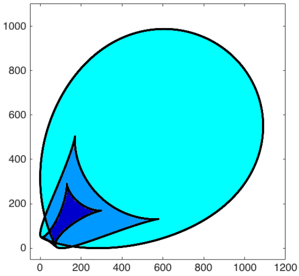
$x^{(5)}$



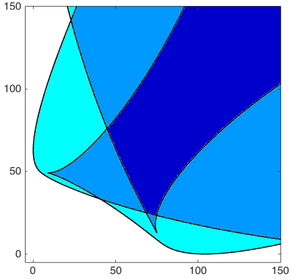
$x^{(6)}$

The 6 solutions to $F(p, \phi; c^*) = 0$.

3RPR mechanism



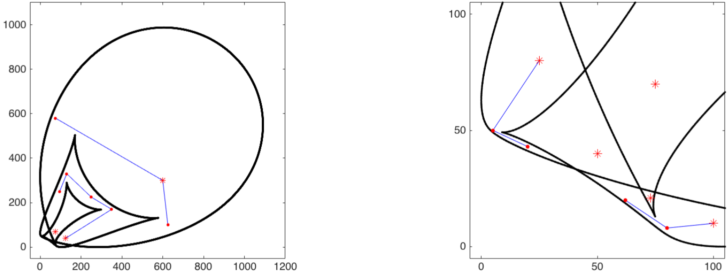
(a)



(b)

Regions of the parameter space $c = (c_1, c_2)$ colored by the number of real solutions where (a) is the full view and (b) is a zoomed in view of the lower left corner. The navy blue region has 6 real solutions, the grey blue region has 4 real solutions, the baby blue region has 2 real solutions, and the white region has 0 real solutions.

3RPR mechanism

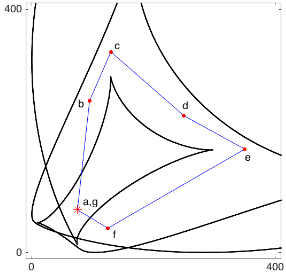


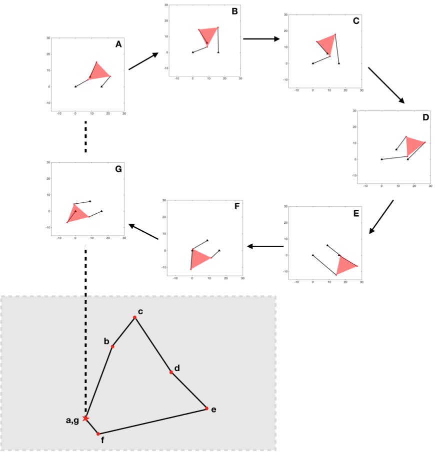
(a) (b)

Selected marked points (red star) in each connected component and all intermediary points (red dot) in (a) full view and (b) zoomed in view of the lower left corner.

3RPR mechanism

Illustration of a nonsingular assembly mode change between $x^{(4)}$ and $x^{(5)}$.





- $\mathcal{G}_1$
  - $\{1\},\{2\},\{3\} \mapsto \{\{1\},\{2\},\{3\}\}$
  - $\{4\},\{5\},\{6\} \mapsto \{\{4\},\{5\},\{6\}\}$
- $\mathcal{G}_2$
  - $\begin{array}{c}\{1,4\},\{1,5\},\{1,6\},\{2,5\},\\ \{2,6\},\{3,4\},\{3,5\}\end{array} \mapsto \left\{\begin{array}{c}\{1,4\},\{1,5\},\{1,6\},\{2,5\},\\ \{2,6\},\{3,4\},\{3,5\}\end{array}\right\}$
  - $\{1,3\},\{2,3\} \mapsto \{\{1,3\},\{2,3\}\}$
  - $\{4,6\},\{5,6\} \mapsto \{\{4,6\},\{5,6\}\}$
  - $\{q_1,q_2\} \mapsto \{\{q_1,q_2\}\}$ for all others
- $\mathcal{G}_3$
  - $\{1,4,6\},\{1,5,6\},\{2,5,6\} \mapsto \{\{1,4,6\},\{1,5,6\},\{2,5,6\}\}$
  - $\{1,3,6\},\{2,3,6\} \mapsto \{\{1,3,6\},\{2,3,6\}\}$
  - $\{3,4,6\},\{3,5,6\} \mapsto \{\{3,4,6\},\{3,5,6\}\}$
  - $\{q_1,q_2,q_3\} \mapsto \{\{q_1,q_2,q_3\}\}$ for all others
- $\mathcal{G}_4$
  - $\{1,3,4,6\},\{1,3,5,6\},\{2,3,5,6\} \mapsto \{\{1,3,4,6\},\{1,3,5,6\},\{2,3,5,6\}\}$
  - $\{q_1,q_2,q_3,q_4\} \mapsto \{\{q_1,q_2,q_3,q_4\}\}$ for all others

Note: $\mathcal{G}_5$ and $\mathcal{G}_6$ are trivial. Thus, the real monodromy group is trivial. However, the complex monodromy group is $\mathcal{S}_6$.

Summary

An extension of the complex monodromy group to the real numbers can be defined in two ways:

- real monodromy group
  - very restrictive and often trivial
- real monodromy structure
  - gives tiered information about the structure of real solutions

Real monodromy structure $\mathcal{G}_1$ describes nonsingular assembly mode changes and can be useful for calibration.

Future work – computing real monodromy structure for:

- Stewart-Gough platforms
- chemical reaction networks

Thank you!

Real monodromy action $\rightarrow$ doi.org/10.1016/j.amc.2019.124983