**Didier Henrion**

# Globally optimal solution to inverse kinematics of a serial manipulator with 7 degrees of freedom

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## Outline

1. Inverse kinematics as a POP

2. Solving the POP with the Lasserre hierarchy

3. Symbolic reduction of the POP

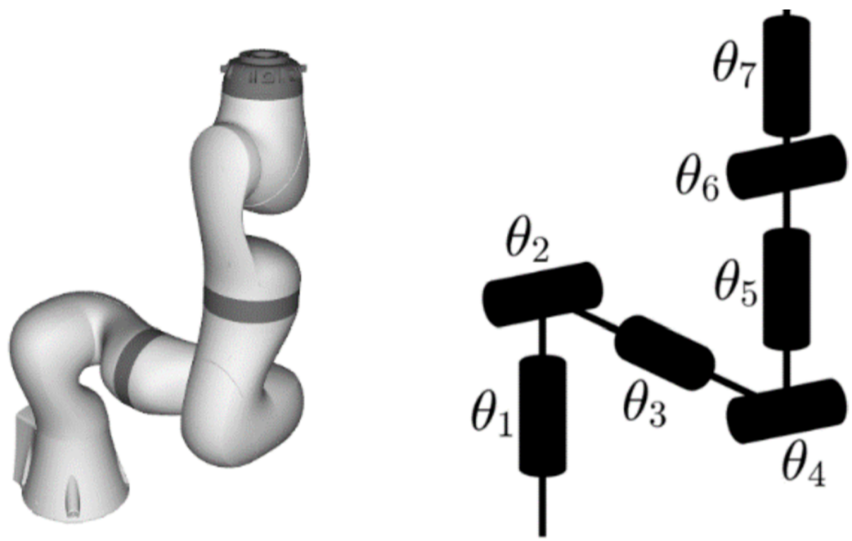
4. Experiments

## 1. Inverse kinematics as a POP

**Inverse kinematics** consists of finding robot control parameters bringing it into a desired position under configuration constraints

For 6 degrees of freedom (DOF), reduces to finding (finitely many real) solutions of a system of polynomial equations

For $> 6$ DOF, optimization over algebraic variety $=$ **polynomial optimization problem** (POP)



Forward kinematics: from end to base, the transformation matrix

$$M = M_1(\theta_1)M_2(\theta_2)\cdots M_7(\theta_7) \in \mathbb{R}^{4\times 4}$$

contains the 3d position vector and a rotation matrix, where each factor $M_i(\theta_i) \in \mathbb{R}^{4\times 4}$ is a trigonometric function of the joint angle $\theta_i$ for $i = 1, \ldots, 7$

Inverse kinematics: given matrix $M$, find angles $\theta_i$, $i = 1, \ldots, 7$

With each angle $\theta_i$ parameterized by cosine $c_i$ and sine $s_i$, this is a system of 12 polynomial equations of degree 7 in 14 variables

The degree can be reduced to 4 by observing that

$$M_3(\theta_3)M_4(\theta_4)M_5(\theta_5) = M_2^{-1}(\theta_2)M_1^{-1}(\theta_1)M M_7^{-1}(\theta_7)M_6^{-1}(\theta_6)$$

Among the infinite number of solutions, we minimize

$$\sum_{i=1}^{7} w_i(\theta_i - \widehat{\theta}_i \bmod \pi)$$

subject to bound constraints

$$\theta_i^{\text{low}} \leq \theta_i \leq \theta_i^{\text{high}},\ i = 1, \ldots, 7$$

Inverse kinematics as a non-convex POP

$$\begin{array}{ll}\min\limits_{\substack{c\in\mathbb{R}^7\\ s\in\mathbb{R}^7}} & \sum_{i=1}^{7} 2w_i(1 - c_i \cos\widehat{\theta}_i - s_i \sin\widehat{\theta}_i) \\ \text{s.t.} & p_j(c, s) = 0, \quad j = 1,\dots,12 \\ & q_i(c_i, s_i) = 1 - {c_i}^2 - {s_i}^2 = 0 \\ & -(c_i + 1)\tan\theta_i^{\text{low}}/2 + s_i \geq 0 \\ & (c_i + 1)\tan\theta_i^{\text{high}}/2 - s_i \geq 0, \quad i = 1,\dots,7\end{array}$$

where polynomials $p_j$ have degree 4 in 14 variables

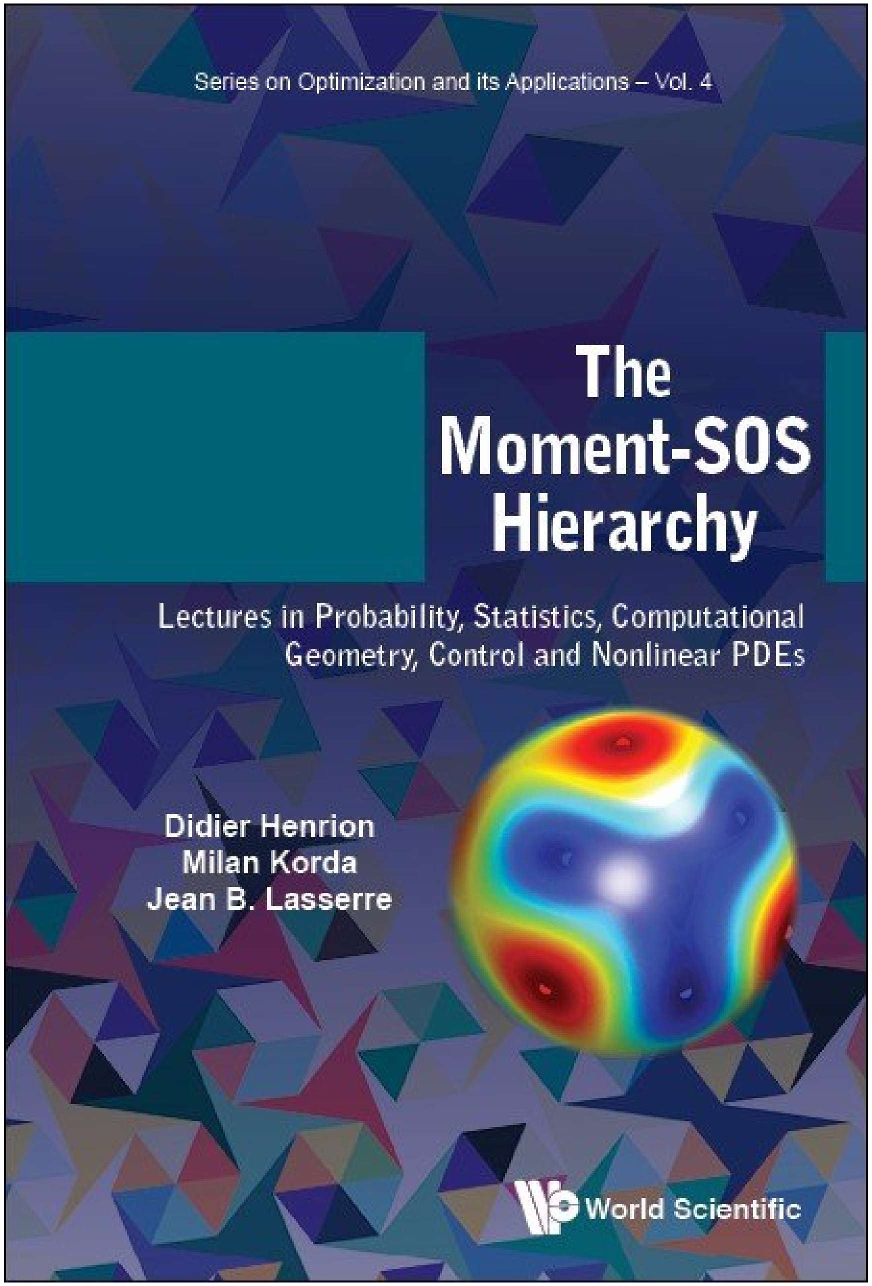
## 2. Solving the POP with the Lasserre hierarchy

To solve the POP numerically, we use the moment-SOS aka Lasserre hierarchy

Based on the duality between the cone of positive polynomials and moments and their sum of squares (SOS) approximations

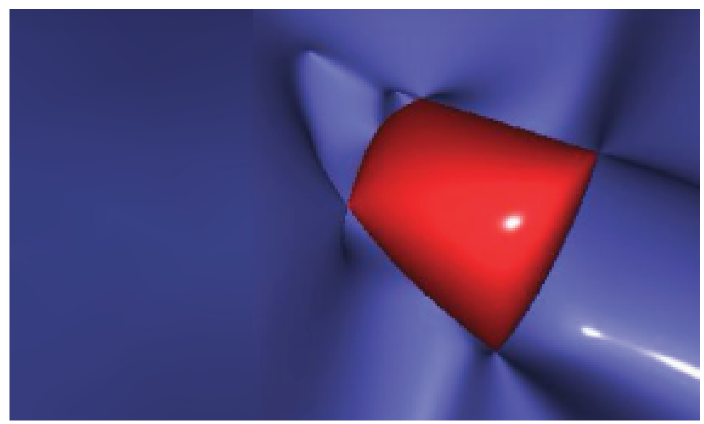
It consists of solving a family of convex (semidefinite) relaxations of increasing size, indexed by a relaxation order $r$, until some rank conditions certify global optimality

homepages.laas.fr/henrion/software/gloptipoly




Poema
Polynomial Optimization, Efficiency through Moments and Algebra

poema-network.eu



The size of the convex relaxations grows in $O(r^n)$ where $n$ is the number of variables of the POP

Typically global optimality is certified for small values of $r$, yet...

.. the 7DOF IK POP of degree 4 has $n = 14$ variables and globally optimality is not always certified for $r = 2$

For $r = 3$ the relaxation contains a dense semidefinite block of size 3060 in 38760 variables - **too large !**

# 3. Symbolic reduction of the POP

**Theorem:** the ideal generated by the degree 4 kinematic polynomials $p_j$ and the degree 2 trigonometric polynomials $q_i$ can be generated by a set of degree 2 polynomials

**Proof:** It is computational:

- choose a generic instance of a serial manipulator and a pose

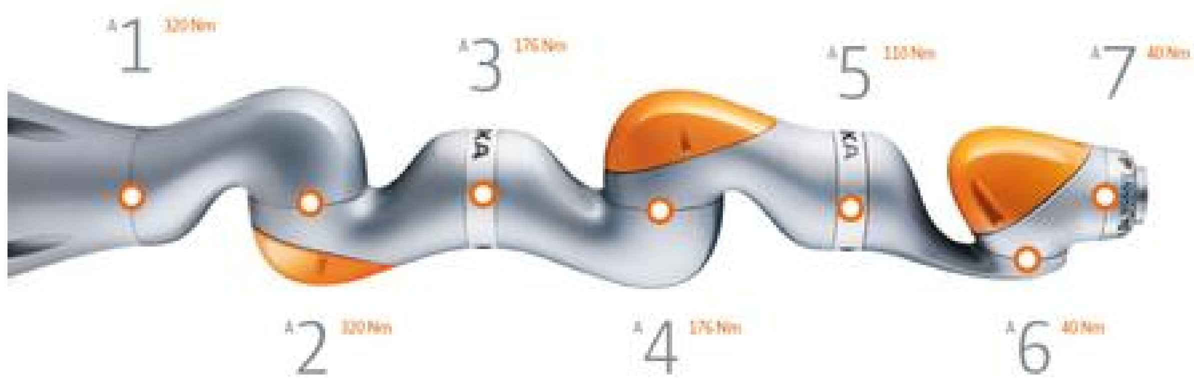
- generate a Gröbner basis $B_4$ of the ideal spanned by $p_j, q_i$

- generate a Gröbner basis $B_2$ of the ideal generated by degree 2 polynomials in $B_4$

- verify that the ideal generated by $B_2$ and $B_4$ is the same

```
# compute the reduced Groebner basis from pj and qi polynomials
    (in variables of eq)
G := Basis(eq, tdeg(op(indets(eq)))):
# select degree two polynomials from the basis and compute a new
     reduced Groebner basis
idxDegTwo := SearchAll(2, map(degree, G)):
eqPrime := G[[idxDegTwo]]:
GPrime := Basis(eqPrime, tdeg(op(indets(eq)))):
# compare the two bases
evalb(G = GPrime);
                              True
```

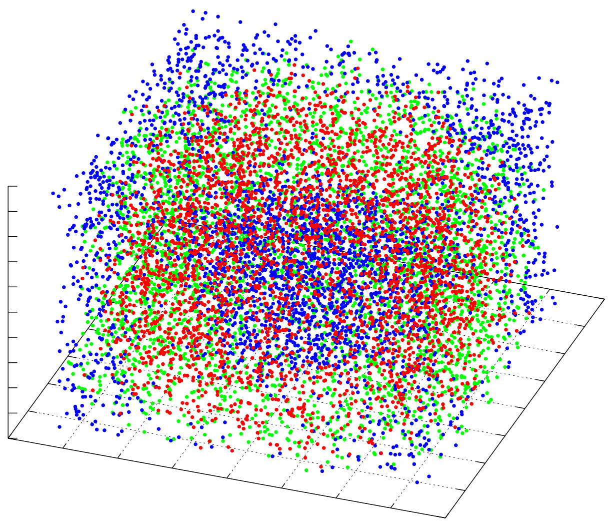
# 4. Experiments


A1 320 Nm
A2 320 Nm
A3 176 Nm
A4 176 Nm
A5 110 Nm
A6 40 Nm
A7 40 Nm

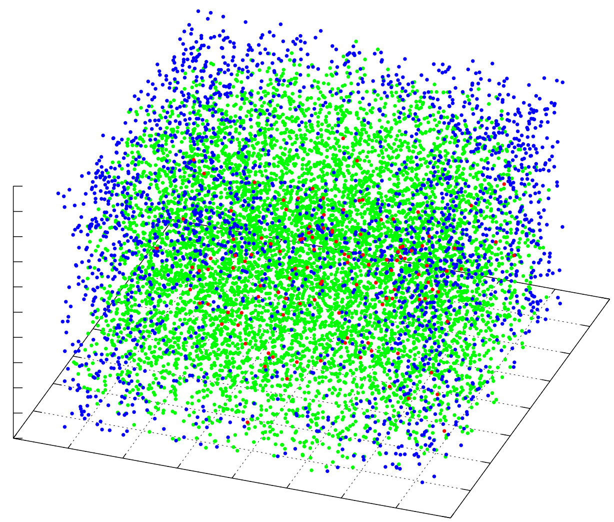
SDP relaxation of order $r = 3$ solves globally the reduced POP for almost all instances

Original POP with SDP relaxation of size 3060 in 38760 variables

Reduced POP with SDP relaxation of size 120 in 3060 variables

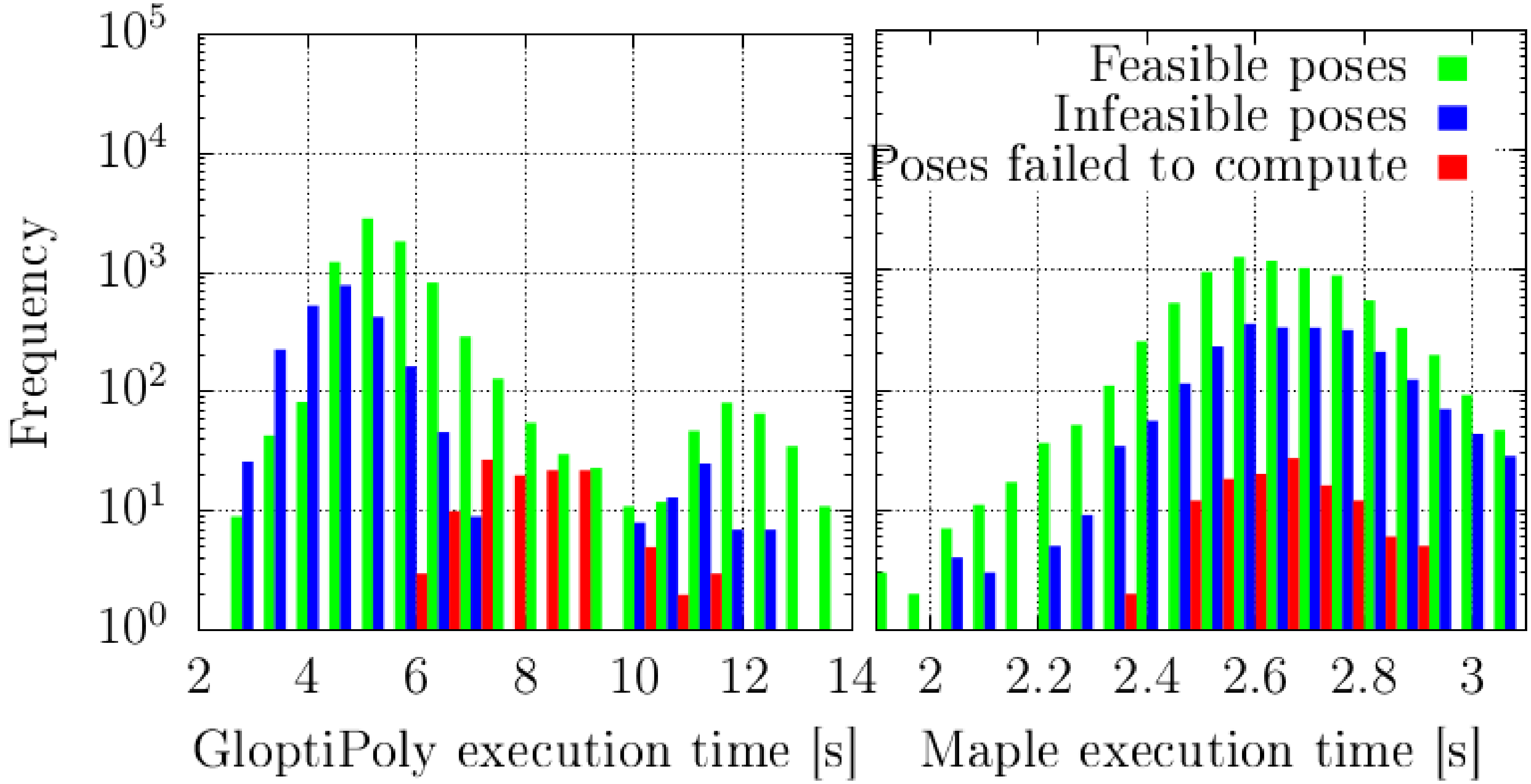


Original POP
32.4% failure rate



Reduced POP
1.2% failure rate

| | Execution time [s] | | Median error | | % of failed |
|---|---|---|---|---|---|
| | **Reduction step** | **GloptiPoly** | **Tran.** [mm] | **Rot.** [deg] | **poses** |
| Deg. 4 pol. | — | 21.3 | $3.92 \cdot 10^{-4}$ | $6.11 \cdot 10^{-5}$ | 32.4 % |
| Deg. 2 pol. | 2.7 | 5.6 | $7.27 \cdot 10^{-5}$ | $5.59 \cdot 10^{-3}$ | 1.2 % |



Frequency
10^0
10^1
10^2
10^3
10^4
10^5
2 4 6 8 10 12 14
GloptiPoly execution time [s]
2 2.2 2.4 2.6 2.8 3
Maple execution time [s]
Feasible poses
Infeasible poses
Poses failed to compute

Our contributions:

- inverse kinematics formulated as a degree 4 POP
- symbolic **reduction** to degree 2 POP
- experimental **validation** with the Lasserre hierarchy

Extensions:

- certify (numerically) **infeasibility** using convex duality
- exploit the structure of the POP to prove **exactness** of the 2nd relaxation of the Lasserre hierarchy

Thanks for your attention !

For more details please refer to

P. Trutman, M. Safey El Din, D. Henrion, T. Pajdla.
Globally Optimal Solution to Inverse Kinematics of 7DOF Serial Manipulator. hal-02905816, arXiv:2007.12550, 2020

people.ciirc.cvut.cz/∼trutmpav

github.com/PavelTrutman/Global-7DOF-IKT