Quantum no-signalling correlations and non-local games

Ivan Todorov

with M. Brannan, S. Harris, L. Turowska

1 June 2021
49th Canadian Operator Symposium

Outline

(1) Non-local games and no-signalling

(2) Why quantise correlations?

(3) Quantum no-signalling correlations

(4) Synchronicity – classical and quantum

(5) Back to non-local games

Non-local games: definition

A non-local game is a tuple $\mathcal{G} = (X, Y, A, B, \lambda)$, where X, Y, A and B are finite sets and $\lambda : X \times Y \times A \times B \to \{0, 1\}$.

Intuition:

- X and Y are input sets of questions for players Alice and Bob.
- A and B are output sets of answers for Alice and Bob.
- λ is the rule predicate of the game.
- Alice and Bob play cooperatively against a Verifier.
- In one round, Alice receives $x \in X$, replies $a \in A$.
- Bob receives $y \in Y$, replies $b \in B$.
- They win if $\lambda(x, y, a, b) = 1$, and lose otherwise.
- Alice and Bob know the rules λ.
- They are not allowed to communicate during the game.
- They may agree on a joint strategy beforehand.

Non-local games: strategies

A deterministic strategy: $f : X \to A, \;\; g : Y \to B$.

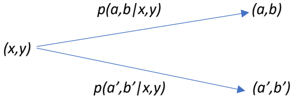
It is a perfect strategy if $\lambda(x, y, f(x), g(y)) = 1$ for all $x \in X$, $y \in Y$.

More generally: The players possess a set $\{(f_i, g_i)\}_{i=1}^k$ of deterministic strategies and employ randomness to choose which one to apply, according to a probability distribution $(\lambda_i)_{i=1}^k$.

> Non-deterministic strategies: to the same input (x, y) the players may reply in different rounds with different outputs (a, b) and (a', b').

$\rightsquigarrow$ A probabilistic strategy: $\{(p(a, b|x, y))_{(a,b)\in A\times B} : (x, y) \in X \times Y\}$, where $p(\cdot, \cdot|x, y)$: a probability distribution for each (x, y).

Intuition: $p(a, b|x, y)$ is the probability of reply (a, b) to a question (x, y).



p a perfect strategy if $\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0$.

No-signalling strategies

Strategy: $p = \{p(\cdot,\cdot|x,y) : x \in X, y \in Y\}$.

Requirement: The strategies Alice and Bob possess do not allow communication between them.

Formally: Well-defined marginals:

$$p(a|x) = \sum_{b\in B} p(a,b|x,y'), \quad p(b|y) = \sum_{a\in A} p(a,b|x',y).$$

p is a no-signalling (NS) correlation if it satisfies these conditions.

Notation: $\mathcal{C}_{\rm ns}$.

Intuitively: No-signalling means that Alice's behaviour is independent from Bob's.

Local correlations: p is a convex combination of correlations

$$p(a,b|x,y) = p_1(a|x)p_2(b|y),$$

for some probability distributions $p_1(\cdot|x)$, $p_2(\cdot|y)$.

Notation: $\mathcal{C}_{\rm loc}$.

The Mermin-Peres magic square game

Goal: Filling in a 3×3 matrix with entries ± 1.

- X the set of rows, Y the set of columns;
- Given a row i, Alice replies $(a_{i,1}, a_{i,2}, a_{i,3}) \in \{-1,+1\}^3$ s.t. $a_{i,1}a_{i,2}a_{i,3} = 1$;
- Given a column j, Bob replies $(b_{1,j}, b_{2,j}, b_{3,j}) \in \{-1,+1\}$ s.t. $b_{1,j}b_{2,j}b_{3,j} = -1$;
- $a_{i,j} = b_{i,j}$.

There exist selfadjoint operators $X_{i,j}$ with spectrum in $\{-1,1\}$ s.t.

$$\prod_{j=1}^{3} X_{i,j} = I \text{ and } \prod_{i=1}^{3} X_{i,j} = -I.$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\rightsquigarrow$

$I \otimes \sigma_z$	$\sigma_z \otimes I$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes I$	$I \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$-\sigma_x \otimes \sigma_z$	$-\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

The correlation chain

Recall: POVM – a family $(E_a)_{a\in A}$ of pos. op. on a Hilbert space H with $\sum_{a\in A} E_a = I$.

A correlation $p = (p(a,b|x,y))$ is called

- quantum if
$$p(a,b|x,y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi\rangle,$$
where $(E_{x,a})_{a\in A}$, $(F_{y,b})_{b\in B}$ fin. dim. POVM's. Notation: $\mathcal{C}_{\mathrm{q}}$.

  Note: $\sum_{a\in A} E_{x,a} = I$ for all x $\Rightarrow$ no-signalling.
- approximately quantum if $p \in \overline{\mathcal{C}_{\mathrm{q}}}$. Notation: $\mathcal{C}_{\mathrm{qa}}$.
- quantum commuting if $p(a,b|x,y) := \langle E_{x,a}F_{y,b}\xi, \xi\rangle$. Notation: $\mathcal{C}_{\mathrm{qc}}$.

Strict inclusions: $\mathcal{C}_{\mathrm{loc}} \subset \mathcal{C}_{\mathrm{q}} \subset \mathcal{C}_{\mathrm{qa}} \subset \mathcal{C}_{\mathrm{qc}} \subset \mathcal{C}_{\mathrm{ns}}$

- $\mathcal{C}_{\mathrm{loc}} \neq \mathcal{C}_{\mathrm{q}}$: Bell's Theorem, 1964
- $\mathcal{C}_{\mathrm{q}} \neq \mathcal{C}_{\mathrm{qa}}$: Slofstra, 2017
- $\mathcal{C}_{\mathrm{qa}} \neq \mathcal{C}_{\mathrm{qc}}$: Ji-Natarajan-Vidick-Wright-Yuen, 2020 (Tsirelson's Problem)
- $\rightsquigarrow$ Connes Embedding Problem has a negative answer: Junge-Navascues-Palazuelos-Perez-Garcia-Scholz-Werner, 2011 + Ozawa, 2013

The universal C*-algebra of a family of PVM's

X, A fin. sets, $M_X = \mathcal{L}(\mathbb{C}^X)$, $\mathcal{D}_X \subseteq M_X$ the diagonal, $\epsilon_{x,x'}$ the mat. units

$$\mathcal{A}_{X,A} := \mathcal{D}_A *_1 \cdots *_1 \mathcal{D}_A \quad (|X| \text{ terms}):$$

generators $e_{x,a} = e_{x,a}^2 = e_{x,a}^*$ with $\sum_{a\in A} e_{x,a} = 1$, $\forall x \in X$.

Universal property: For every family $(E_{x,a})_{a\in A}$, $x \in X$, of PVM's on H, there exists a (unique) *-representation $\pi : \mathcal{A}_{X,A} \to \mathcal{B}(H)$ with

$$\pi(e_{x,a}) = E_{x,a}, \quad x \in X, a \in A.$$

For a functional $s : \mathcal{A}_{X,A} \otimes \mathcal{A}_{Y,B} \to \mathbb{C}$, write

$$p_s(a,b|x,y) = s(e_{x,a} \otimes e_{y,b}).$$

Description in terms of states (Junge et al + Ozawa, Lupini et al)

- $\mathcal{C}_{\mathrm{qc}} = \{p_s : s \text{ a state on } \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}\}$;
- $\mathcal{C}_{\mathrm{qa}} = \{p_s : s \text{ a state on } \mathcal{A}_{X,A} \otimes_{\min} \mathcal{A}_{Y,B}\}$;
- $\mathcal{C}_{\mathrm{ns}} = \{p_s : s \text{ a state on } \mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}\}$ ($\mathcal{S}_{X,A} = \mathrm{sp}\{e_{x,a}\}$ oper. subsys. of $\mathcal{A}_{X,A}$.)

Quantum no-signalling correlations (as opposed to no-signalling quantum correlations)

Motivating question I: Suppose the game has quantum inputs/outputs.
What kind of strategies can be used?

Note : The players remain classical, Alice and Bob.

- Regev-Vidick, 2015: Quantum XOR games
- Cooney-Junge-Palazuelos-Pérez-García, 2015: Rank one quantum games

Setup: Alice and Bob receive (fin. dim.) quantum states as inputs and apply quantum operations to produce quantum states as outputs.

Note: A classical input (x, y) gives rise to the state $\epsilon_{x,x} \otimes \epsilon_{y,y} \in \mathcal{D}_X \otimes \mathcal{D}_Y$.

$\rightsquigarrow$ Question: Is there a correlation chain on the level of quantum strategies?

Preferably looking like $\mathcal{Q}_{\rm loc} \subset \mathcal{Q}_{\rm q} \subset \mathcal{Q}_{\rm qa} \subset \mathcal{Q}_{\rm qc} \subset \mathcal{Q}_{\rm ns}$?

Partial answers:

- Quantum strategies for quantum games (i.e. $\mathcal{Q}_{\rm q}$) used in the games above.
- No-signalling strategies (i.e. $\mathcal{Q}_{\rm ns}$) used by Duan-Winter as resource in zero-error quantum information transmission.

$\rightsquigarrow$ Motivating question II: Perhaps a suitable (simpler and genuinely) quantum game can disprove Tsirelson-Connes?

Classical correlations as channels $\rightsquigarrow$ quantum correlations

A NS correlation $p = (p(a,b|x,y))$ gives rise to an information channel (positive trace preserving map) $\mathcal{N}_p : \mathcal{D}_X \otimes \mathcal{D}_Y \to \mathcal{D}_A \otimes \mathcal{D}_B$:

$$\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,b} p(a,b|x,y)\epsilon_{a,a} \otimes \epsilon_{b,b}$$

$\rightsquigarrow$ Quantum no-signalling (QNS) correlations: quantum channels (completely positive trace preserving maps) $\Gamma : M_X \otimes M_Y \to M_A \otimes M_B$,

- $\rho \in M_{XY}$, $\mathrm{Tr}_X \rho = 0 \implies \mathrm{Tr}_A \Gamma(\rho) = 0$; $(M_{XY} := M_X \otimes M_Y)$
- $\rho \in M_{XY}$, $\mathrm{Tr}_Y \rho = 0 \implies \mathrm{Tr}_B \Gamma(\rho) = 0$. (Duan-Winter, 2016)

Notation: $\mathcal{Q}_{\mathrm{ns}}$.

Local QNS correlations: $\Phi : M_X \to M_A$, $\Psi : M_Y \to M_B$

$\rightsquigarrow \Gamma = \Phi \otimes \Psi$ and their convex combinations.

Notation: $\mathcal{Q}_{\mathrm{loc}}$.

Stochastic operator matrices as quantum POVM's

$E = (E_{x,x',a,a'})_{x,x',a,a'} \in M_{XA} \otimes \mathcal{B}(H)$ is a stochastic operator matrix if $\mathrm{Tr}_A E = I_X \otimes I_H$:

$$\sum_{a \in A} E_{x,x',a,a'} = \delta_{x,x'} I_H, \quad x, x' \in X.$$

A family $\{(E_{x,a})_{a\in A} : x \in X\}$ of POVM's gives rise to $E = \sum_{x,a} \epsilon_{x,x} \otimes \epsilon_{a,a} \otimes E_{x,a}$.

A QNS correlation $\Gamma : M_{XY} \to M_{AB}$ is called

- quantum if
  $$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} \langle (E_{x,x',a,x'} \otimes F_{y,y',b,b'})\xi, \xi \rangle \epsilon_{a,a'} \otimes \epsilon_{b,b'},$$
  with E and F are fin. dim. stochastic operator matrices. Notation: $\mathcal{Q}_{\mathrm{q}}$
- approximately quantum if $\Gamma \in \overline{\mathcal{Q}_{\mathrm{q}}}$. Notation: $\mathcal{C}_{\mathrm{qa}}$
- quantum commuting when $E_{x,x',a,x'} \otimes F_{y,y',b,b'}$ is replaced by $E_{x,x',a,x'} F_{y,y',b,b'}$. Notation: $\mathcal{Q}_{\mathrm{qc}}$

$$\mathcal{Q}_{\mathrm{loc}} \subset \mathcal{Q}_{\mathrm{q}} \subset \mathcal{Q}_{\mathrm{qa}} \subset \mathcal{Q}_{\mathrm{qc}} \subset \mathcal{Q}_{\mathrm{ns}}$$

The universal C*-algebra of a family of PVM's

Let $\mathcal{V}$ be the universal TRO of an isometry $V = (v_{a,x})_{a,x}$ ($x \in X$, $a \in A$).
$\rightsquigarrow \mathcal{C}_{X,A} = \overline{\text{span}(\mathcal{V}^*\mathcal{V})}$.

Generators: $e_{x,x',a,a'} = v^*_{a,x} v_{a',x'}$; $E = (e_{x,x',a,a'})$ is a universal stoch. op. matrix

Universal property: For every stochastic op. matrix $(E_{x,x',a,a'})$ on H there exists a (unique) *-representation $\pi : \mathcal{C}_{X,A} \to \mathcal{B}(H)$ with

$$\pi(e_{x,x',a,a'}) = E_{x,x',a,a'}, \quad x, x' \in X, a, a' \in A.$$

Key: The map $\Phi : \epsilon_{a,a'} \to E_{x,x',a,a'}$, $M_A \to M_X \otimes \mathcal{B}(H)$ is unital and cp $\rightsquigarrow$ an application of Stinespring's Theorem

A functional $s : \mathcal{C}_{X,A} \otimes \mathcal{C}_{Y,B} \to \mathbb{C}$ induces

$$\Gamma_s(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} s(e_{x,x',a,a'} \otimes f_{y,y',b,b'})\epsilon_{a,a'} \otimes \epsilon_{b,b'}.$$

Description in terms of states (T-Turowska, 2020)

- $\mathcal{Q}_{\rm qc} = \{\Gamma_s : s \text{ a state on } \mathcal{C}_{X,A} \otimes_{\max} \mathcal{C}_{Y,B}\}$;
- $\mathcal{C}_{\rm qa} = \{\Gamma_s : s \text{ a state on } \mathcal{C}_{X,A} \otimes_{\min} \mathcal{C}_{Y,B}\}$;
- $\mathcal{C}_{\rm ns} = \{\Gamma_s : s \text{ a state on } \mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B}\}$ ($\mathcal{T}_{X,A} = {\rm sp}\{e_{x,x',a,a'}\}$ oper.subsys.of $\mathcal{C}_{X,A}$.)

Semi-classical cases

Quantum-to-classical correlations: Brannan-Ganesan-Harris, 2020.

Strategies: $(p(a, b|\varphi))$, where $a \in A$, $b \in B$, $\varphi \in M_{XY}$ state.

$\rightsquigarrow$ POVM's $(P_a)_{a\in A}$ and $(Q_b)_{b\in B}$.

Classical-to-quantum correlations: T-Turowska, 2020.

$$\mathcal{E} : \mathcal{D}_{XY} \to M_{AB} \ \rightsquigarrow \ \text{states } \sigma_{x,y} := \mathcal{E}(\epsilon_{x,x} \otimes \epsilon_{y,y}).$$

No-signalling: $\mathrm{Tr}_A \sigma_{x,y} = \mathrm{Tr}_A \sigma_{x',y}$ and $\mathrm{Tr}_B \sigma_{x,y} = \mathrm{Tr}_A \sigma_{x,y'}$

Universal algebra: $\mathcal{B}_{X,A} = M_A *_1 \cdots *_1 M_A$ ($|X|$ times)

$\rightsquigarrow$ similar descriptions in terms of states on $\mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}$ etc.

Classical	Classical-to-quantum	Quantum
$\mathcal{N} : \mathcal{D}_{XY} \to \mathcal{D}_{AB}$	$\mathcal{E} : \mathcal{D}_{XY} \to M_{AB}$	$\Gamma : M_{XY} \to M_{AB}$
$\mathcal{A}_{X,A} = C^*(e_{x,a})$	$\mathcal{B}_{X,A} = C^*(e_{x,a,a'})$	$\mathcal{C}_{X,A} = C^*(e_{x,x',a,a'})$
POVM's $(E_{x,a})_{a\in A}$	$E_x \in M_A^+$ with $\mathrm{Tr}_A E_x = I$	$E = (E_{x,x',a,a'})$ stochastic

Synchronicity

A NS correlation $p = ((p(a,b|x,y))$ is synchronous if $Y = X$, $B = A$ and

$$a \neq b \implies p(a,b|x,x) = 0, \quad x \in X.$$

Operational meaning: Alice and Bob are required to produce identical outputs if their inputs are identical.

Example: The graph colouring game for a graph G on a vertex set X: synchronicity +

$$x \sim y \implies p(a,a|x,y) = 0, \quad a \in A.$$

Note: p synchronous iff

$$\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{x,x}) \leq J_A^{\rm cl} := \sum_{a \in A} \epsilon_{a,a} \otimes \epsilon_{a,a}.$$

Description in terms of traces

- $p \in \mathcal{C}_{\rm qc}$ synchronous iff $\exists$ trace $\tau : \mathcal{A}_{X,A} \to \mathbb{C}$ s.t. $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$;

  Paulsen-Severini-Stahlke-T-Winter, 2016
- $p \in \mathcal{C}_{\rm qa}$ synchronous iff $\exists$ amenable trace $\tau : \mathcal{A}_{X,A} \to \mathbb{C}$ s.t. $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$.

  Kim-Paulsen-Schafhauser, 2018

Concurrency

I. A classical-to-quantum $\mathcal{E} : \mathcal{D}_{XX} \to M_{AA}$ concurrent if

$$\mathcal{E}(\epsilon_{x,x} \otimes \epsilon_{x,x}) = J_A := \frac{1}{|A|} \sum_{a,b} \epsilon_{a,b} \otimes \epsilon_{a,b}, \quad x \in X.$$

J_A: the maximally entangled state in M_{AA}

Description in terms of traces (Brannan-Harris-T-Turowska, 2021)

- $\mathcal{E} \in \mathcal{CQ}_{\mathrm{qc}}$ concurrent iff $\exists$ trace $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ s.t.

$$\mathcal{E}(\epsilon_{x,x} \otimes \epsilon_{y,y}) = [\tau(e_{x,a,a'} e_{y,b',b})]_{a,a',b,b'};$$

- $\mathcal{E} \in \mathcal{CQ}_{\mathrm{qa}}$ concurrent iff $\exists$ amenable trace $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ with this property.

II. A quantum $\Gamma : M_{XX} \to M_{AA}$ concurrent if $\Gamma(J_X) = J_A$.

Description in terms of traces (ditto, Bochniak-Kasprzak-Sołtan, 2021)

- $\Gamma \in \mathcal{Q}_{\mathrm{qc}}$ concurrent $\Rightarrow$ $\exists$ trace $\tau : \mathcal{C}_{X,A} \to \mathbb{C}$ s.t.

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = [\tau(e_{x,x',a,a'} e_{y,y',b',b})]_{a,a',b,b'};$$

- $\Gamma \in \mathcal{Q}_{\mathrm{qa}}$ concurrent iff τ can be chosen amenable.

Concurrency

Ingredients:

- $\Gamma = \Gamma_s$ for a state $s : \mathcal{C}_{X,A} \otimes_{\max} \mathcal{C}_{X,A} \to \mathbb{C}$;
- Γ concurrent: $\sum_{x,y} s(e_{x,y,a,b} \otimes f_{x,y,a,b}) = \frac{|X|}{|A|}$, $a, b \in A$.
- $\rightsquigarrow s(u(e_{x,y,a,b} \otimes 1 - 1 \otimes f_{y,x,b,a})) = 0 \;\; \forall\, u$;
- $\rightsquigarrow s(ze_{x,y,a,b} \otimes 1) = s(z \otimes f_{y,x,b,a}) = s(e_{x,y,a,b}z \otimes 1)$;
- $\rightsquigarrow z \to s(z \otimes 1)$ a trace.

Conversely: Assume $X = A$.

Brown algebra: C*-alg. $\mathcal{B}_X$ generated by the entries of a unitary $(u_{a,x})_{a,x}$.

$$\rightsquigarrow p_{x,x',a,a'} = u^*_{a,x} u_{a',x'} \;\; \rightsquigarrow \;\; \mathcal{C}_X := C^*(p_{x,x',a,a'})$$

Description in terms of traces (ditto)

- $\Gamma \in \mathcal{Q}_{\mathrm{qc}}$ concurrent iff $\exists$ trace $\tau : \mathcal{C}_X \to \mathbb{C}$ s.t.
$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = [\tau(p_{x,x',a,a'} p_{y,y',b',b})]_{a,a',b,b'}.$$
- $\Gamma \in \mathcal{Q}_{\mathrm{q}}$ concurrent iff τ factors through a fin. dim. rep.

Open for the class $\mathcal{Q}_{\mathrm{qa}}$.

Key steps:

- If $\mathcal{J} \subseteq \mathcal{C}_{X,A}$ generated by $\sum_{x \in X} e_{y,x,b,a} e_{x,y,a,b} - e_{y,y,b,b}$, $y, a, b \in X$, then $\mathcal{C}_{X,A}/\mathcal{J} = \mathcal{C}_X$.
- In the converse direction, the fact that Γ is a channel forces $\Gamma(J_X) = J_X$.

Rule functions of quantum non-local games

Recall: The rule function of a classical game $\lambda : X \times Y \times A \times B \to \{0,1\}$.

Equivalently: $\varphi_\lambda : \mathrm{Proj}(\mathcal{D}_{XY}) \to \mathrm{Proj}(\mathcal{D}_{AB})$ given by

$$\varphi_\lambda\Big(\sum_{(x,y)\in\kappa} \epsilon_{x,x}\otimes\epsilon_{y,y}\Big)=\sum\{\epsilon_{a,a}\otimes\epsilon_{b,b} : \lambda(x,y,a,b) = 1 \text{ for some } (x,y) \in \kappa\}.$$

$\rightsquigarrow$ The rules λ can be identified with $\vee$-preserving 0-preserving maps

$$\varphi : \mathrm{Proj}(\mathcal{D}_{XY}) \to \mathrm{Proj}(\mathcal{D}_{AB}).$$

Note: Work of Erdos, 1984, preceded by Loginov-Shulman, 1973, on reflexivity.

Proposed definition, T-Turowska, 2020

The rule function of a quantum non-local game is a $\vee$-preserving 0-preserving map

$$\varphi : \mathrm{Proj}(M_{XY}) \to \mathrm{Proj}(M_{AB}).$$

A perfect strategy for φ is a QNS correlation $\Gamma : M_{XY} \to M_{AB}$ s.t.

$$\langle \Gamma(P), \varphi(P)^\perp \rangle = 0.$$

Note: Quantum XOR games fit into this framework, as do some rank one quantum games (Crann-Levene-Turowska-Winter).

Quantum orthogonal rank

Recall: The graph colouring game for a graph G with vertex set X: $Y = X$, $A = B$,

- $x = y \;\Rightarrow\; a = b$
- $x \sim y \;\Rightarrow\; a \neq b$.

p is a perfect strategy for the colouring game iff

$$x \sim y \Longrightarrow \langle \mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}), J_A \rangle = 0.$$

Quantise: Using classical-to-quantum correlations

$\rightsquigarrow$ no-signalling assignment $(x, y) \to \sigma_{x,y}$ s.t.

$$x \sim y \Longrightarrow \langle \sigma_{x,y}, J_A \rangle = 0, \quad x, y \text{ vertices of } G.$$

Aim: Choose smallest A for which this is possible with an element of a specific class $\{\mathrm{loc}, \mathrm{q}, \mathrm{qc}\} \rightsquigarrow \xi_{\mathrm{loc}}(G)$, $\xi_{\mathrm{q}}(G)$, $\xi_{\mathrm{qc}}(G)$

Note: A quantum rule $\varphi : \mathrm{Proj}(M_{XY}) \to \mathrm{Proj}(M_{AB})$ gives rise to a an ideal $\mathcal{J}_\varphi \subseteq \mathcal{C}_{X,A}$

$\rightsquigarrow$ C*-algebra $C^*(\varphi)$ of the quantum game $\varphi \rightsquigarrow \xi_{\mathrm{C}^*}(G)$, $\xi_{\mathrm{alg}}(G)$.

Quantum orthogonal rank

Theorem (T-Turowska, 2020)

$\xi_{\rm loc}(G)$ coincides with the orthogonal rank $\xi(G)$ of G.

$\xi(G)$: the smallest k s.t. $\exists$ unit vectors $\xi_x \in \mathbb{C}^k$, $x \in X$, with $x \sim y \Rightarrow \langle \xi_x, \xi_y \rangle = 0$.

Key: Observe that the tracial local strategies are convex comb. of $\Phi \otimes \Phi^\sharp$. Use the monotonicity of the trace to reduce the colouring to one of the form $\sigma_x \otimes \sigma_x^{\rm t}$. Use the diagonal form of σ_x and monotonicity to get the vectors ξ_x.

Theorem (T-Turowska, 2020, Brannan-Harris-T-Turowska, 2021)

$\xi_{\rm q}(G) \geq \sqrt{\frac{|X|}{\theta(G)}}$, and $\xi_{\rm q}(K_{d^2}) = \xi_{\rm C^*}(K_{d^2}) = d$.

$\theta(G)$: Lovász number of G; K_m: the complete graph on m vertices.

Key:

- Superdense coding: unitaries $U_{a,a'} \in M_A$ with $\mathrm{Tr}(U_{a,a'} U_{b,b'}^*) = \delta_{(a,a'),(b,b')}$.
- $A = \mathbb{Z}_d = \{0, 1, \dots, d-1\}$, $X = A \times A$, ζ a primitive d-th root of unity.
- For $x = (a', b')$ and $y = (a'', b'') \in X$, let $\xi_{x,y} = \frac{1}{\sqrt{d}} \zeta^{b''(a''-a')} \sum_{l=0}^{d-1} \zeta^{(b''-b')l} e_l \otimes e_{l-a'+a''}$ and $\sigma_{x,y} = \xi_{x,y}\xi_{x,y}^*$.
- $E_{x,z,z'} = \zeta^{(z'-z)b'} \epsilon_{z-a',z'-a'} \in M_d$.

THANK YOU VERY MUCH!