Distance between reproducing kernel Hilbert spaces and geometry of finite sets in the unit ball

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Overview

- Based on : **Distance between reproducing kernel Hilbert spaces and geometry of finite sets in the unit ball** (with Danny Ofek and Orr Shalit), Journal of Mathematical Analysis and Applications (2021)

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- $\mathcal{H}$ - a reproducing kernel Hilbert space (RKHS) on a set X;
- K - the reproducing kernel for $\mathcal{H}$;
- $\text{Mult}\,\mathcal{H} = \{f : X \to \mathbb{C} | fh \in \mathcal{H} \text{ for every } h \in \mathcal{H}\}$ - the **multiplier algebra** of $\mathcal{H}$.

- Goal : Present, quantitatively, the relationship between an RKHS, its multiplier algebra, and the geometry of the underlying set.

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- We will restrict our attention to finite-dimensional complete Pick spaces.

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Notions of isomorphism for RKHSs

For $i = 1, 2$, let $\mathcal{H}_i$ be reproducing kernel Hilbert spaces respectively on sets X_i with reproducing kernels $K_i(x, y) = k^i_y(x)$.

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Definition

An **isomorphism of RKHSs** from $\mathcal{H}_1$ to $\mathcal{H}_2$ (or simply, an **RKHS isomorphism**) is a bijective bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ defined by

$$T(k_x^1) = \lambda_x k_{F(x)}^2 \quad \text{for all} \quad x \in X_1,$$

where $x \mapsto \lambda_x$ is a nowhere-vanishing complex valued function defined on X_1 and $F : X_1 \to X_2$ is a bijection.

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Notions of isomorphism for multiplier algebras

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$$(T : \mathcal{H}_1 \to \mathcal{H}_2) \text{ induces } (M_f \mapsto (T^*)^{-1} M_f T^*)$$

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Even more, it suffices to study quotients of H^2_d that arise from multiplier varieties, (i.e. quotients of H^2_d that arise from X that is the joint zero set of an ideal of multipliers in $\mathcal{M}_d$).

Theorem (Davidson-Ramsey-Shalit ; 2015 (see also Rochberg ; 2019))

Let $V, W \subseteq \mathbb{B}_d$ be multiplier varieties. Then TFSAE.

- *$\mathcal{M}_V$ is isometrically isomorphic to $\mathcal{M}_W$.*
- *V and W are congruent (i.e., $\exists$ a biholo auto $\Phi \in \mathrm{Aut}(\mathbb{B}_d)$ s.t $\Phi(V) = W$)*
- *$\mathcal{H}_V$ is isometrically isomorphic as an RKHS to $\mathcal{H}_W$.*
- *$\mathcal{M}_V$ is completely isometrically isomorphic to $\mathcal{M}_W$.*

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Our problem

Question : Let X_1 and X_2 be finite sets of points in $\mathbb{B}_d$ considered as metric spaces.

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- And conversely, if the function spaces are “close" to being isometrically isomorphic in some sense, are the sets then “close" to be being biholomorphic images one of the other ?
- What about the multiplier algebras ? How are they determined by the underlying sets and their geometry ?

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Distance between sets

Suppose that the unit ball $\mathbb{B}_d$ is equipped with the pseudohyperbolic metric ρ_{ph}.

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- The **automorphism invariant symmetric distance** between subsets of $\mathbb{B}_d$.

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Banach-Mazur analog

For the spaces :

- The reproducing kernel Banach-Mazur distance ρ_{RK} - how far two spaces are from being isometrically isomorphic as RKHSs.

$$\rho_{RK}(\mathcal{H}_1, \mathcal{H}_2) = \log\left(\inf\left\{\|T\|\|T^{-1}\| : T : \mathcal{H}_1 \to \mathcal{H}_2 \text{ is an RKHS isomorphism}\right\}\right)$$

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For the multiplier algebras :

- Multiplier Banach-Mazur distance ρ_M - how far two multiplier alebras are from being completely isometrically isomorphic as multiplier algebras

$$\rho_M(\mathcal{M}_1, \mathcal{M}_2) = \log\left(\inf\left\{\|\varphi\|_{cb}\|\varphi^{-1}\|_{cb} : \varphi : \mathcal{M}_1 \to \mathcal{M}_2 \text{ a mult alg isom}\right\}\right).$$

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Main result

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Theorem (Ofek-P.-Shalit ; 2021)

Fix $n \geq 2$. Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{B}_d$ and let $Y^{(k)} = \{y_1^{(k)}, \ldots, y_n^{(k)}\}$ be a sequence of subsets of $\mathbb{B}_d$. Let $\tilde{\rho}_s$ be the automorphism invariant symmetric distance induced by the pseudohyperbolic metric ρ_{ph}. Put $\mathcal{H} = H_d^2\big|_X$, $\mathcal{M} = \mathrm{Mult}(\mathcal{H})$, $\mathcal{H}_k = H_d^2\big|_{Y^{(k)}}$ and $\mathcal{M}_k = \mathrm{Mult}(\mathcal{H}_k)$. Then, the following statements are equivalent.

1. $\tilde{\rho}_s(X, Y^{(k)}) \xrightarrow{k\to\infty} 0.$
2. $\rho_{RK}(\mathcal{H}, \mathcal{H}_k) \xrightarrow{k\to\infty} 0.$
3. $\rho_M(\mathcal{M}, \mathcal{M}_k) \xrightarrow{k\to\infty} 0.$

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On the proof

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- $1 \implies 2$; $\delta_{RK}(\mathcal{H}_X, \mathcal{H}_Y) \leq \left(1 + \frac{4nr(1-r^2)^{-2}}{\min\{\lambda_{\min}(A), \lambda_{\min}(B)\}} \rho_s(X, Y)\right)^2$,

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$$\min_{W \in \mathcal{U}(d)} \|A - WB\|_2^2 \leq d(2\|A\|_2 \varepsilon^{1/2} + \varepsilon).$$

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- $2 \implies 3$; $\rho_M(\mathcal{M}_1, \mathcal{M}_2) \leq \rho_{RK}(\mathcal{H}_1, \mathcal{H}_2)^2$.

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A few words on the significance of our result

- Let d be a positive integer and $t \in (0, \infty)$. Then there exists an RKHS H_d^t on $\mathbb{B}_d$ with the reproducing kernel

$$K(x, y) = \frac{1}{(1 - \langle x, y \rangle)^t}.$$

- When $d = 1$, H_d^t is a weighted Hardy space.

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Theorem (Ofek-Sofer ; 2021)

Let $\mathcal{H} = H_d^t$.

- *If $t \in (0, 2]$, then for every $A, B \subseteq \mathbb{B}_d$, we have*

$$\mathcal{H}_A \text{ is isometrically isomorphic as an RKHS to } \mathcal{H}_B \iff A \text{ is congruent to } B$$

- *If $t \in (2, \infty)$, then there exist non congruent subsets of $\mathbb{B}_d$ that yield isometrically isomorphic RKHSs.*

Thank you

Thank you for your time.

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