Traces on locally compact groups

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49th Canadian Operator Symposium
University of Guelph
June 3, 2021

Positive definite functions and traces

G – topological group

$u : G \to \mathbb{C}$ is positive definite if

$$\sum_{i,j=1}^{n} \overline{\lambda_i} u(s_j^{-1} s_i) \lambda_j \geq 0 \quad \begin{array}{l} \forall \lambda_1, \ldots, \lambda_n \in \mathbb{C} \\ s_1, \ldots, s_n \in G, n \in \mathbb{N} \end{array}$$

i.e. $[u(s_j^{-1} s_i)] \in \mathrm{M}_n(\mathbb{C})$ is positive.

Note: $|u(s)| \leq u(e)$ as $\begin{bmatrix} u(e) & u(s) \\ u(s^{-1}) & u(e) \end{bmatrix} \geq 0.$

Basic spaces

$$\mathrm{P}(G) = \{u \in \mathcal{C}(G) : u \text{ pos. def.}\}$$
$$\mathrm{P}_1(G) = \{u \in \mathrm{P}(G) : u(e) = 1\}$$
$$\mathrm{T}(G) = \{u \in \mathrm{P}_1(G) : u(st) = u(ts),\ \forall s, t \in G\}$$

Locally compact case

G – locally compact group, admits left Haar measure.
Hence $\mathcal{C}_c(G) \subseteq \mathrm{L}^1(G)$ are involutive algebras

$$f * g(t) = \int_G f(s)g(s^{-1}t)\, ds, \quad f^*(t) = \overline{f(t^{-1})}\Delta(t^{-1})$$

Universal C*-algebra:

$$\mathrm{C}^*(G) = \overline{\mathcal{C}_c(G)} = \overline{\mathrm{L}^1(G)}$$

(completion w.r.t. largest C*-norm); $\lambda : \mathrm{L}^1(G) \to \mathcal{B}(\mathrm{L}^2(G))$, $\lambda(f)h = f * h$ insures a norm.

States and tracial states

$$\mathrm{P}_1(G) \cong \mathrm{S}(\mathrm{C}^*(G)), \quad \langle f, u \rangle = \int_G f(s)u(s)\, ds$$

$$\mathrm{T}(G) \cong \mathrm{T}(\mathrm{C}^*(G))$$

Reduced traces

$\lambda : G \to \mathrm{U}(\mathrm{L}^2(G))$ left reg. rep'n,

$$\mathrm{P}_r(G) = \overline{\{\langle \lambda(\cdot)h|h\rangle : h \in \mathrm{L}^2(G)\}}^{uc} = \{\langle \pi(\cdot)\xi|\xi\rangle : \xi \in \mathcal{H}_\pi, \pi \prec \lambda\}$$

('$\prec$'="weakly contained")

$$\mathrm{C}^*_r(G) = \overline{\lambda(\mathrm{L}^1(G))} = \overline{\lambda(\mathcal{C}_c(G))} \subseteq \mathcal{B}(\mathrm{L}^2(G))$$

Reduced traces:

$$\mathrm{T}_r(G) = \mathrm{P}_r \cap \mathrm{T}(G)$$

States and tracial states

$$\mathrm{P}_{1,r}(G) \cong \mathrm{S}(\mathrm{C}^*_r(G)), \quad \mathrm{T}_r(G) \cong \mathrm{T}(\mathrm{C}^*_r(G))$$

Some notes on discrete groups

Γ – discrete group

At least two traces guaranteed:

$$1,\ 1_{\{e\}} = \langle \lambda(\cdot)\delta_e | \delta_e \rangle \in \mathrm{T}(\Gamma).$$

If $N \lhd \Gamma$ (amenable), q.l.r.r. $\lambda_{\Gamma/N} : \Gamma \to \mathrm{U}(\ell^2(\Gamma/N))$

$$1_N = \langle \lambda_{\Gamma/N}(\cdot)\delta_{eN} | \delta_{eN} \rangle \in \mathrm{T}(\Gamma) \text{ (even in } \mathrm{T}_r(\Gamma))$$

[Trucker-Drob '12] If $\mu \in \mathrm{Prob}(\mathrm{Sub}(\Gamma))$ invariant, where $\mathrm{Sub}(\Gamma) \tilde{\subset} \{0,1\}^\Gamma$, then

$$\int_{\mathrm{Sub}(\Gamma)} 1_H \, d\mu(H) \in \mathrm{T}(\Gamma) \text{ (in } \mathrm{T}_r(\Gamma) \text{ if } \mathrm{supp}\mu \subseteq \mathrm{Sub}_{am}(\Gamma)\ (*)).$$

[Bader–Duchense–Lécureux '16] $(*) \Rightarrow \mathrm{supp}\mu \subseteq \mathrm{Sub}(AR(\Gamma))$.

Theorem [Breuillard–Kalantar–Kennedy–Ozawa '17]

$\mathrm{T}_r(\Gamma) = \{1_{\{e\}}\} \ \Leftrightarrow\ \{e\}$ only normal amenable subgroup

Return to l.c. groups

$$\begin{aligned} G \text{ abelian } &\Rightarrow \mathrm{T}(G) = \mathrm{P}_1(G) \cong \mathrm{Prob}(\widehat{G}) \\ G \text{ compact } &\Rightarrow \mathrm{T}(G) = \overline{\mathrm{conv}}\left\{ \frac{1}{d_\pi}\chi_\pi : \pi \in \widehat{G} \right\} \end{aligned}$$

Hence we expect many traces if either

G/G' is infinite, or

G admits infinitely many f.d. (irreducible) rep'ns

Tracial kernel

Proposition (GNS)

If $u \in \mathrm{P}_1(G)$, there is w.o.-continuous $\pi_u : G \to \mathrm{U}(\mathcal{H}_u)$, $\xi \in \mathcal{H}_u$ unit, so $u = \langle \pi_u(\cdot)\xi | \xi \rangle$.

Proposition

(i) $u \in \mathrm{P}_1(G) \Rightarrow N_u = \{s \in G : u(s) = 1\}$ closed subgroup.

(ii) $u \in \mathrm{T}(G) \Rightarrow N_u = \ker \pi_u \lhd G$.

Idea. Use uniform convexity of $\mathcal{H}_u$.

Tracial kernel:

$$N_{\mathrm{Tr}} = N_{\mathrm{Tr}}(G) = \bigcap_{u \in \mathrm{T}(G)} N_u.$$

Then $\mathrm{T}(G) = \mathrm{T}(G/N_{\mathrm{Tr}}) \circ q_{N_{\mathrm{Tr}}}$.

Related kernels

$$N_{\mathrm{MAP}} = \bigcap_{N \in \mathcal{N}_{\mathrm{MAP}}} N, \quad \mathcal{N}_{\mathrm{MAP}} = \{N \lhd G : \text{closed}, \ G/N \in [\mathrm{MAP}]\}$$

– von Neumann kernel, MAP = maximally almost periodic

$$N_{\mathrm{SIN}} = \bigcap_{N \in \mathcal{N}_{\mathrm{SIN}}} N, \quad \mathcal{N}_{\mathrm{SIN}} = \{N \lhd G : \text{closed}, \ G/N \in [\mathrm{SIN}]\}$$

– SIN = small invariant neighbourhood

Easy to show that $G/N_{\mathrm{MAP}} \in [\mathrm{MAP}]$.

Problem: G/N_{SIN} may not live in $[\mathrm{SIN}]$.

If $N \in \mathcal{N}_{\mathrm{SIN}}$ then $(G/N_{\mathrm{SIN}})/(N/N_{\mathrm{SIN}}) \cong G/N \in [\mathrm{SIN}]$.

I.e. G/N_{SIN} is residually-SIN; write $G/N_{\mathrm{SIN}} \in [\mathrm{RSIN}]$.

An inclusion of classes

Theorem

$[\mathrm{MAP}] \subset [\mathrm{RSIN}]$.

Idea. $G_0 \cong V \times K$. If $\pi : G \to \mathrm{U}(d)$, can show that $\ker \pi \in \mathcal{N}_{\mathrm{SIN}}$.

Corollary

$G \in [\mathrm{MAP}]$ and tot. disc'd $\Rightarrow$ G is residually discrete.
$G \in [\mathrm{MAP}]$, tot. disc'd, & com. gen. $\Rightarrow$ G is residually finite.

Idea. There is a compact open subgroup $L \subseteq \ker \pi$.
If compactly generated, use [Mal'cev '40].

Inclusions of classes

$$\begin{tikzcd}
{[\mathrm{K}]} \arrow[r, "\mathbb{Z}"] \arrow[dr, "\mathbb{R}"'] & {[\mathrm{RK}]} \arrow[r, "\mathbb{Q},\mathbb{Q}_p"] & {[\mathrm{MAP}] = [\mathrm{RMAP}]} \arrow[d] \\
{[\mathrm{D}]} \arrow[r] & {[\mathrm{SIN}]} \arrow[r] & {[\mathrm{RSIN}]}
\end{tikzcd}$$

Examples. $\Gamma \in [\mathrm{D}]$, $\alpha \in F \subseteq \mathrm{Aut}(\Gamma)$ finite, $\{\alpha(s)s^{-1} : s \in \Gamma\}$ infinite

$$G = \underbrace{\Gamma^{\oplus \mathbb{N}}}_{\text{discrete}} \rtimes \underbrace{F^{\mathbb{N}}}_{\text{compact}} \in [\mathrm{RSIN}] \setminus [\mathrm{SIN}].$$

(i) [Murakami '50] $\Gamma = \mathbb{Z}$, $F = \langle \text{inversion} \rangle$: $G \in [\mathrm{RK}] \setminus [\mathrm{SIN}]$

(ii) $\Gamma = \mathrm{SL}_2(\mathbb{Q})$, $F = \left\langle \mathrm{Ad} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$:

$G \in [\mathrm{RSIN}] \setminus ([\mathrm{SIN}] \cup [\mathrm{MAP}])$ [von Neumann–Wigner '40]

Back to tracial kernel

Proposition

$N_{\mathrm{Tr}} \subseteq N_{\mathrm{SIN}} \subseteq N_{\mathrm{MAP}}$

Idea. $N_{\mathrm{SIN}} \subseteq N_{\mathrm{MAP}}$ since $[\mathrm{MAP}] \subset [\mathrm{RSIN}]$.
If $N \in \mathcal{N}_{\mathrm{SIN}}$, U rel. com. inv. nbhd. of eN in G/N, then

$$u_U = \frac{1}{m(U)} \langle \lambda_{G/N}(\cdot) 1_U | 1_U \rangle \in \mathrm{T}(G).$$

Remark. If there is non-open $N \in \mathcal{N}_{\mathrm{SIN}}$, or infinite index $N \in \mathcal{N}_{\mathrm{MAP}}$, then $\dim \mathrm{T}(G) = \infty$.

Question. Is there G s.t. $N_{\mathrm{Tr}} \subsetneq N_{\mathrm{SIN}}$?

The role of compact generation

Proposition

Let H be a top'l group, with a family $\mathcal{B}$ of open invariant subsets with $\bigcap_{B\in\mathcal{B}} B = \{e\}$. Suppose G is com. gen., & admits cts. injective hom'm $\varphi : G \to H$. Then $G \in [\mathrm{SIN}]$.

(i) G com. gen. $\Rightarrow N_{\mathrm{Tr}} = N_{\mathrm{SIN}}$.

(ii) G com. gen. in $[\mathrm{RSIN}] \Rightarrow G \in [\mathrm{SIN}]$.

Essential idea in [Hofmann-Mostert '63], used to show com. gen. MAP is SIN.
(i) $\varphi : G/N_u \to \pi_u(G) \subseteq \mathrm{U}(\mathcal{H}_u)$, s.o.t.=w.o.t.
(ii) $\varphi(s) = (q_N(s)) : G \to \prod_{N\in\mathcal{N}_{\mathrm{SIN}}} G/N$

$$[\mathrm{K}] \xrightarrow{\mathbb{Z},\mathbb{R}} [\mathrm{RK}] \xrightarrow{\mathbb{R}^2\rtimes C_3} [\mathrm{MAP}] \xrightarrow{BS} [\mathrm{SIN}]$$

Connected group G

[Freudenthal '36/Weil '40] G connected: then

$$G \in [\mathrm{SIN}] \quad \Leftrightarrow \quad G \in [\mathrm{MAP}] \quad \Leftrightarrow \quad G = V \times K.$$

Consequence. $N_{\mathrm{Tr}} = N_{\mathrm{MAP}}$

Lie case. R – solvable radical, $S = S_c S_{nc}$ – Levi complement

[Shtern '10] $N_{\mathrm{MAP}} = \overline{[R, G] S_{nc}}$

Consequence. $G/N_{\mathrm{Tr}} = V \times K$, V quot. of max'l vector subgp. of $G/\overline{[R, G]}$, semisimple part of K loc. isom. to S_c.

If G is simply connected, then $V = G/[R, G]$ and $K = S_c$.

Existence of reduced traces I

Lemma

G totally disconnected and $\mathrm{SIN} \Leftrightarrow G$ pro-discrete.

Theorem [Forrest–S.–Wiersma '17]

TFAE for G compactly generated:

(i) $\mathrm{T}_r(G) \neq \varnothing$

(ii) $\mathcal{N}_{\mathrm{SIN}}$ admits an amenable element

(iii) G admits an open normal amenable subgroup

(iv) $N^r_{\mathrm{Tr}} = \bigcap_{u \in \mathrm{T}_r(G)} N_u$ ($= G$ if $\mathrm{T}_r(G) = \varnothing$) is amenable

(i) $\Rightarrow$ (ii) $u \in \mathrm{T}_r(G)$, $1 \in \mathrm{P}_r(G)|_{N_u} \Rightarrow N_u$ amen.; $G/N_u \in [\mathrm{SIN}]$.
(ii) $\Rightarrow$ (iii) $N \in \mathcal{N}_{\mathrm{SIN}}$ amenable, $\widetilde{(G/N)/(G/N)_0}$ has open compact normal L, $V \times K \cong (G/N)_0 \to \widetilde{L} \to L$, $\widetilde{L} \subset G/N$ open normal.

Hence if G connected: $\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow G$ amenable.

Existence of reduced traces II

$AR(G)$ – amenable radical

Theorem [Kennedy–Raum '17]

$\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow AR(G)$ open. In this case

$$\mathrm{T}_r(G) \cong \{u \in \mathrm{T}(AR(G)) : u(srs^{-1}) = u(r)\ \forall s \in G, r \in AR(G)\}.$$

Hence

$$\mathrm{T}_r(G) \cong \left\{ u \in \mathrm{T}(AR(G)/N^r_{\mathrm{Tr}}) : \begin{array}{c} u(srs^{-1}N^r_{\mathrm{Tr}}) = u(rN^r_{\mathrm{Tr}}) \\ \forall s \in G, r \in AR(G) \end{array} \right\}.$$

E.g. [Powers '75] $\mathrm{T}_r(\mathrm{F}_2) = \{1_{\{e\}}\}$, i.e. $AR(\mathrm{F}_2) = \{e\}$.
[Pashcke-Salinas '79] free products
[de la Harpe '83, Bekka-Cowling-de la Harpe '94] $\mathrm{PSL}_n(\mathbb{Z})$ etc.

Groups with(out) unique reduced trace

Theorem

If A is non-discrete abelian, Γ is fin. gen. disc., $\Gamma \curvearrowright A$ irred.
then $N^r_{\mathrm{Tr}}(A \rtimes \Gamma) \cap A$ either $\{e\}$ or A;
former case $\Leftrightarrow$ $\Gamma \curvearrowright A$ factors through compact $K \curvearrowright A$.

Idea. A either connected or tot. disc'd., $A \rtimes \Gamma$ com. gen.
If $N^r_{\mathrm{Tr}}(A \rtimes \Gamma) \cap A = \{e\}$, Ascoli Thm. of [Grosser-Moskowitz '71].

Proposition

K – compact, Γ – disc., $\Gamma \curvearrowright K$
$N^r_{\mathrm{Tr}}(K \rtimes \Gamma) \cap K = K$ unless $X_K = \{\frac{1}{d_\pi}\chi_\pi : \pi \in \widehat{G}\}$ admits non-trivial finite Γ-orbit.

Remark. semi-direct products can be replaced by extensions, e.g. $A \to G \to \Gamma$

Groups with(out) unique reduced trace: Examples

Below, $G = A \rtimes \Gamma$ or $K \rtimes \Gamma$.

(i) $\Gamma = \mathrm{F}_2 \subset \mathrm{SO}(3)$ dense, $A = \mathbb{R}^3$.
$AR(G) = \mathbb{R}^3$, $N^r_{\mathrm{Tr}}(G) = \{(0,0)\}$, $\mathrm{T}_r(G) \cong \mathrm{Prob}(\mathbb{R}^3)^{\mathrm{SO}(3)}$.

(ii) $\Gamma = \mathrm{F}_2$, $q : \mathrm{F}_2 \to \mathbb{Z}^2$, $\eta : \mathbb{Z}^2 \to \mathbb{R}$, $\eta(m,n) = m + \sqrt{2}n$
$M \in \mathrm{GL}_2(\mathbb{R})$, $\sigma(M) \subset \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{T})$.
$s \in \mathrm{F}_2$, $v \in A = \mathbb{R}^2$, $s \cdot v = \exp(iM\eta \circ q(s))v$

(iii) $d \geq 3$ odd, $\Gamma = \mathrm{SL}_d(\mathbb{Z}) = \mathrm{PSL}_d(\mathbb{Z})$, $K = \mathbb{T}^d$; $A = \mathbb{R}^d$.

(iv) $AR(\Gamma) = \{e\}$, $L \supsetneq \{e\}$ compact, $K = L^\Gamma$, shift action.

(v) $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[1/p])$, $A = \mathrm{M}_2(\mathbb{Q}_p)/\mathbb{Q}_p \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\sigma \cdot x = \sigma x \sigma^T$.

(vi) $d \geq 3$ odd, $\Gamma = \mathrm{SL}_d(\mathbb{Z})$, $A = \mathbb{Q}_p^d$, $\mathrm{T}_r(G) \cong \mathrm{Prob}(\mathbb{Q}_p^d)^{\mathrm{SL}_d(\mathbb{Q}_p)}$

(ii)-(v) satisfy

$$AR(G) = N^r_{\mathrm{Tr}}(G) = H,\ \mathrm{T}_r(G) = \{1_H\},\ H = A \text{ or } K$$

Amenable traces

$u \in \mathrm{T}(G)$, define $\tilde{u} : G \times G \to \mathbb{C}$ by

$$\tilde{u}(s,t) = u(st^{-1})$$

Fact. $\tilde{u} \in \mathrm{P}_1(G \times G)$.

Let $\mathrm{P}_{\mathsf{min}}(G \times G) = \overline{\mathrm{cone}\{u \times v : u, v \in \mathrm{P}(G)\}}^{uc}$, so

$$\mathrm{P}_{\mathsf{min},1}(G \times G) \cong \mathrm{S}(\mathrm{C}^*(G) \otimes_{\mathsf{min}} \mathrm{C}^*(G)).$$

We let the set of <u>amenable traces</u>:

$$\mathrm{T}_{\mathrm{am}}(G) = \{u \in \mathrm{T}(G) : \tilde{u} \in \mathrm{P}_{\mathsf{min}}(G \times G)\}$$

and the <u>amenable tracial kernel</u>:

$$N_{\mathrm{amTr}} = \bigcap_{u \in \mathrm{T}_{\mathrm{amTr}}(G)} N_u.$$

Containments of kernels

(i) $N_{\mathrm{amTr}} \subseteq N_{\mathrm{MAP}}$.

(ii) If $\mathrm{C}^*(G)$ is nuclear (G almost connected; amenable; or type I, e.g. $G = \mathrm{G}(\Bbbk)$ [Bekka-Echterhoff '20]) then $\mathrm{T}(G) = \mathrm{T}_{\mathrm{am}}(G)$ and $N_{\mathrm{Tr}} = N_{\mathrm{amTr}}$.

(iii) G almost connected, $G/N_{\mathrm{Tr}} = V \rtimes K$, $K \curvearrowright V$ factors through finite [Grosser-Moskowitz '71]. Hence $G/N_{\mathrm{Tr}} \in [\mathrm{MAP}]$, so $N_{\mathrm{Tr}} = N_{\mathrm{MAP}}$.

Theorem

G has property (T) $\Rightarrow N_{\mathrm{amTr}} = N_{\mathrm{MAP}}$.

Idea. [Ozawa '04, Kirchberg '94] (T) and (F) $\Rightarrow$ residually finite.

E.g. F finite simple, $G = F^{\oplus \mathbb{Z}} \rtimes \mathbb{Z}$
amenable so $1_{\{e\}} \in \mathrm{T}_{\mathrm{amTr}}(G)$, i.e. $\{e\} = N_{\mathrm{amTr}}$; $N_{\mathrm{MAP}} = F^{\oplus \mathbb{Z}}$.

Factorization property

$\lambda \cdot \rho : G \times G \to \mathrm{U}(\mathrm{L}^2(G))$, $\lambda \cdot \rho(s,t)h(r) = h(s^{-1}rt)\Delta(t)^{1/2}$

Factorization property (property (F)): for each h in $\mathrm{L}^2(G)$

$$\langle \lambda \cdot \rho(\cdot,\cdot)h|h\rangle \in \mathrm{P}_{\mathsf{min}}(G \times G).$$

I.e. induces representation of $\mathrm{C}^*(G) \otimes_{\mathsf{min}} \mathrm{C}^*(G)$.

Theorem

(i) $G \in [\mathrm{SIN}]$ with property (F) $\Rightarrow N_{\mathrm{amTr}} = \{e\}$

(ii) G tot. disc'd & $N_{\mathrm{amTr}} = \{e\} \Rightarrow$ property (F)

G either almost conn., or tot. disc. w. $N_{\mathrm{amTr}} = \{e\} \Rightarrow$ (F)

Question. Generally, does $N_{\mathrm{amTr}} = \{e\} \Rightarrow$ (F)?

Amenable reduced traces

[Lance '73] Γ disc.: $\mathrm{C}^*_r(\Gamma)$ nuclear $\Leftrightarrow$ Γ amenable.
[Ng '15] $\mathrm{C}^*_r(G)$ nuclear & $\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow G$ amenable.

$$\mathrm{T}_{\mathrm{am},r}(G) = \{u \in \mathrm{T}_r(G) : \tilde{u} \in \mathrm{P}_r(G \times G)\}$$

I.e. $\tilde{u}$ corresponds to trace on $\mathrm{C}^*_r(G) \otimes_{\mathrm{min}} \mathrm{C}^*_r(G) \cong \mathrm{C}^*_r(G \times G)$.

Theorem

TFAE

(i) G amenable
(ii) $\mathrm{T}_{\mathrm{am},r}(G) \neq \varnothing$
(iii) $\mathrm{C}^*_r(G)$ nuclear & $\mathrm{T}_r(G) \neq \varnothing$

(ii) $\Rightarrow$ (i) $u \in \mathrm{T}_{\mathrm{am},r}(G)$, $1 \cong \tilde{u}|_{G_D} \in \mathrm{P}_r(G)$.

Thank-you

Niá:wen!   Miigwech!

(Thank-you, in Mohawk, a Haudenosaunee language; and in Anishnaabemowin of the Mississaugans.)

In Waterloo, we live and work in the traditional territory of the Neutral, Anishinaabeg and Haudenosaunee peoples. We are situated in the Haldimand tract, land promised to the Six Nations (Haudenosaunee), which includes six miles along each side of the Grand River.