Mixed Unitary Rank

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Quantum Channels and Kraus Operators

Definition

A quantum channel is a trace-preserving completely positive linear map, i.e. $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ such that

$$(\mathrm{id}_k \otimes \Phi)(A) \succeq 0$$

for all $M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \ni A \succeq 0$ for any positive integer k, and

$$\mathrm{tr}(\Phi(A)) = \mathrm{tr}(A)$$

for all $A \in M_n(\mathbb{C})$

Theorem (Choi)

A map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is CP if and only if the follow equivalent conditions are satisfied:

- $J(\Phi) := (\mathrm{id}_n \otimes \Phi)(\phi\phi^*) \succeq 0$ *where* $\phi = \sum_i e_i \otimes e_i$
- *There exist Kraus operators* $\{K_i\}_{i=1}^d \subseteq M_{m,n}(\mathbb{C})$ *such that* $\Phi(A) = \sum_{i=1}^d K_i A K_i^*$

Φ is a channel, i.e. trace-preserving, if we further have

$$(\mathrm{id} \otimes \mathrm{tr})(J(\Phi)) = I_n \;\Leftrightarrow\; \sum_i K_i^* K_i = I_n.$$

The minimum number of Kraus operators is the rank of $J(\Phi)$, the Choi rank

Random/Mixed Unitaries

Definition

A quantum channel $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is mixed unitary or random unitary if there exists a probability distribution $(p_1, \cdots, p_m)$ and unitaries $\{U_i\}_{i=1}^m \subseteq \mathcal{U}(n)$ such that

$$\Phi(A) = \sum_{i=1}^{m} p_i U_i A U_i^*$$

i.e., if $\{\sqrt{p_i} U_i\}_{i=1}^m$ are a set of Kraus operators for Φ.

Clearly, $m \geq \mathrm{Choi\ rank}(\Phi)$. The minimal m yielding a mixed unitary description of Φ is the *mixed unitary rank* (m.u. rank)

Question

If $m = \text{m.u. rank}(\Phi)$, $d = \text{Choi rank}(\Phi)$, *other than the trivial* $m \geq d$, *what can we say about the relationship between these two ranks?*

Example (Girard)

Let $\Phi(A) = A \circ C$ where $\circ$ is entrywise Schur product, and C is the correlation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix};$$

then Φ has Choi rank $d = 3$ and m.u. rank $m = 4$.

Given a channel $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ with Kraus operators $\{K_i\}_{i=1}^d$ we define its canonical operator system

$$S_\Phi := \mathrm{span}\{K_i^* K_j\}_{i,j=1}^d$$

which is invariant to the choice of Kraus operators. We also have a map (the adjoint of the complement) $\Phi^C : M_d(\mathbb{C}) \to M_n(\mathbb{C})$

$$\Phi^{C\dagger}(E_{ij}) = K_i^* K_j$$

Proposition (Mixed Unitarity)

A channel Φ is mixed unitary if and only if there exist vectors $\{u_i\}_{i=1}^m \subseteq \mathbb{C}^d$ and positive numbers $\{p_i\}_{i=1}^m$ such that the rank-one matrices satisfy

$$\Phi^{C\dagger}(u_i u_i^*) = p_i I_n$$

and

$$\sum_{i=1}^m u_i u_i^* = I_d$$

The m.u. rank is the smallest m such that a set of such vectors exists

When Φ is a Schur product channel, $\Phi(A) = A \circ C$ for a correlation matrix C, $\Phi^{C\dagger}$ has a particularly nice form. If $\{\overline{w_i}\}_{i=1}^n$ are a set of Gram vectors for C, so that $\overline{c_{ij}} = \langle w_i, w_j \rangle$, then

$$\Phi^{C\dagger}(A) = \mathrm{diag}(\langle w_i, Aw_i \rangle).$$

So Φ is mixed unitary if and only if there exist $\{u_i\}$ with $\sum_i u_i u_i^* = I_d$ such that

$$|\langle w_j, u_i \rangle|^2 = p_i \ \forall j$$

Theorem

Let $\Phi : M_n \to M_n$ have Choi rank d; and the dimension of the operator system S_Φ is $r := \dim(S_\Phi)$.
If Φ is mixed unitary with m.u. rank m, then

$$m \leq d^2 - r + 1.$$

Let $\{u_i\}_{i=1}^m$ be a minimal set of vectors witnessing mixed unitarity of Φ; we can always choose $\{u_i u_i^*\}_{i=1}^m$ to be linearly independent, so $\Phi^{C\dagger}$ is a map whose domain is $M_d(\mathbb{C})$, whose rank is $r := \dim(S_\Phi)$, that sends m linearly independent matrices $\{u_i u_i^*\}$ to the identity. The result is then simply a consequence of the rank-nullity theorem.

Corollary

Since $d \leq m$ we immediately see that we require

$$r \leq d^2 - d + 1$$

and if r achieves this bound, then if Φ is mixed unitary at all, $m = d$

In fact, if Φ is a mixed unitary channel with Choi rank d and $r := \dim(S_\Phi) = d^2 - d + 1$ the choice of unitaries is *unique* (up to ordering and scalar multiplication)

Proposition

Suppose $\widehat{\Phi}$ is mixed unitary with $r = d^2 - d + 1$ (and thus an essentially unique m.u. decomposition) and Ψ is any Choi-rank-1 channel; then the channel

$$\Phi := \Psi \oplus \widehat{\Phi}$$

has
Choi rank $d + 1$ and m.u. rank $2d$

Example (Girard (redux))

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} = 1 \oplus \widehat{C}$$

with

$$\widehat{C} = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

Thus $\Phi(A) = A \circ C = \mathrm{id}_1 \oplus \widehat{\Phi}$ where $\widehat{\Phi}(X) = \widehat{C} \circ X$ has unique m.u. decomposition with m.u. rank $\widehat{m} = \widehat{d} = 2$

Definition

A set of mutually unbiased bases (MUBs) in $\mathbb{C}^d$ is a collection of o.n. bases $\mathcal{B}_i = \{v_j^{(i)}\}_{j=1}^d$, $1 \leq i \leq k$, such that the inner product between any two vectors from different bases has modulus $\frac{1}{\sqrt{d}}$:

$$|\langle v_j^{(i)}, v_l^{(k)} \rangle| = \begin{cases} \delta_{jl} & i = k \\ \frac{1}{\sqrt{d}} & i \neq k \end{cases}$$

It is well-known that the number of bases in a MUB in $\mathbb{C}^d$ is at most $d+1$, and this is achieved for all $d = p^\alpha$, p prime.
In all of what follows, we suppose $\widehat{d} = p^\alpha$ so a set of $\widehat{d}+1$ mutually unbiased bases exists

We form the $\widehat{d}^2 \times \widehat{d}^2$ correlation matrix $\widehat{C}$ as the Gram matrix of the vectors $\{v_j^{(i)}\}_{i,j=1}^{\widehat{d}}$ (notice we do *not* include the vectors from $\mathcal{B}_{\widehat{d}+1}$!); the channel

$$\widehat{\Phi}(A) = A \circ \widehat{C}$$

has Choi rank $\widehat{d}$, and

$$\widehat{r} := \dim(S_{\widehat{\Phi}}) = \widehat{d}^2 - \widehat{d} + 1.$$

What's more, $\widehat{\Phi}$ is mixed unitary (with $\widehat{m} = \widehat{d}$ (necessarily, since $\widehat{r} = \widehat{d}^2 - \widehat{d} + 1$), and thus essentially unique choice of unitaries. Then the remaining basis $\mathcal{B}_{\widehat{d}+1}$ supplies us with $\widehat{d}$ projections $\{v_k^{(\widehat{d}+1)} v_k^{(\widehat{d}+1)*}\}_{k=1}^{\widehat{d}}$ summing to the identity and with

$$\Phi^{C\dagger}(v_k^{(\widehat{d}+1)} v_k^{(\widehat{d}+1)*}) = \frac{1}{\widehat{d}} I_{\widehat{d}^2}.$$

Finally let $C = 1 \oplus \widehat{C}$ and $\Phi(A) = A \oplus C$; this channel then has Choi rank $d = \widehat{d} + 1$ and m.u. rank $m = 2\widehat{d}$, thus the gap between them is $\widehat{d} - 1$, which can be made arbitrarily large by picking different prime powers.

Thank you to the organizers, and to you for your attention