Bistochastic operators and quantum random variables

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I. Preliminaries

The Setting

- X is a locally compact Hausdorff space
- $\mathcal{O}(X)$ is the σ-algebra of Borel sets of X
- $\mathcal{H}$ is a finite or separable Hilbert space
- $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$
- $\mathcal{T}(\mathcal{H})$ is the Banach space of all trace-class operators: all operators in $\mathcal{B}(\mathcal{H})$ which have a finite trace under any orthonormal basis
- The convex subset $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ of all positive, trace-one trace-class operators ρ (called *states* or density operators)

We are interested in positive operator-valued measures $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ and ν-integrable functions $X \to \mathcal{B}(\mathcal{H})$. **Why?** The desire for a notion of an operator-valued averaging, i.e., the quantum expected value of a quantum random variable. To define majorization through the use of bistochastic operators in this setting.

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Operator-valued Measures

Definition

A map $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is an *operator-valued measure (OVM)* if it is ultraweakly countably additive: for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{O}(X)$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)\,,$$

where the convergence on the right side of the equation above is with respect to the ultraweak topology of $\mathcal{B}(\mathcal{H})$, that is,

$$\mathrm{Tr}\left(s\sum_{k=1}^{n}\nu(E_k)\right) \to \mathrm{Tr}\left(s\sum_{k=1}^{\infty}\nu(E_k)\right), \quad \forall s \in \mathcal{S}(\mathcal{H}).$$

Properties

An OVM ν is

(i) *bounded* if $\sup\{\|\nu(E)\| : E \in \mathcal{O}(X)\} < \infty$,

(ii) *positive* if $\nu(E) \in \mathcal{B}(\mathcal{H})_+$, for all $E \in \mathcal{O}(X)$; such an OVM is called a *positive operator-valued measure* (POVM), (Note: A POVM is necessarily bounded)

(iii) *regular* if the induced complex measure $\mathrm{Tr}(\rho\nu(\cdot))$ is regular for every $\rho \in \mathcal{T}(\mathcal{H})$.

(iv) a *positive operator-valued probability measure* or *quantum probability measure* if it is positive and $\nu(X) = I_{\mathcal{H}}$.

We use $\mathrm{POVM}_{\mathcal{H}}(X)$ to refer to the set of all POVMs and and $\mathrm{POVM}^1_{\mathcal{H}}(X)$ to refer to the set of all quantum probability measures.

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Absolute Continuity

Definition

A (classical or operator-valued) measure ω_1 is *absolutely continuous* with respect to either a classical or operator-valued measure ω_2, denoted $\omega_1 \ll_{\mathrm{ac}} \omega_2$, if $\omega_1(E) = 0$ whenever $\omega_2(E) = 0$, where $E \in \mathcal{O}(X)$ (for classical measures, $\mathcal{O}(X)$ is typically denoted by Σ) and 0 is interpreted as either the scalar zero or the zero operator, as applicable.

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$. For a fixed state $\rho \in \mathcal{S}(\mathcal{H})$, the induced complex measure ν_ρ on X is defined by $\nu_\rho(E) = \mathrm{Tr}(\rho\nu(E))$ for all $E \in \mathcal{O}(X)$. Note: ν and ν_ρ are mutually absolutely continuous for any full-rank $\rho \in \mathcal{S}(\mathcal{H})$.

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Building a Radon-Nikodým derivative

Let $\nu_{i,j}$ be the complex measure defined by $\nu_{i,j}(E) = \langle \nu(E)e_j, e_i \rangle$, $E \in \mathcal{O}(X)$, where $\{e_k\}$ form an orthonormal basis for $\mathcal{H}$. Let $\rho \in \mathcal{S}(\mathcal{H})$ be full-rank. Then $\nu_{i,j} \ll_{\mathrm{ac}} \nu_\rho$ and so, by the classical Radon-Nikodým theorem, there is a unique $\frac{d\nu_{i,j}}{d\nu_\rho} \in L_1(X, \nu_\rho)$ such that

$$\nu_{i,j}(E) = \int_E \frac{d\nu_{i,j}}{d\nu_\rho} d\nu_\rho, \ E \in \mathcal{O}(X).$$

One can then define the *Radon-Nikodým derivative* of ν with respect to ν_ρ to be

$$\frac{d\nu}{d\nu_\rho} = \sum_{i,j \geq 1} \frac{d\nu_{i,j}}{d\nu_\rho} \otimes e_{i,j}.$$

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Quantum Random Variables

Definition

An operator-valued function $f : X \to \mathcal{B}(\mathcal{H})$ that is Borel measurable (that is, the associated complex-valued functions $x \to \mathrm{Tr}(sf(x))$ are Borel measurable functions for every state $s \in \mathcal{S}(\mathcal{H})$) is called a *quantum random variable*.

The Radon-Nikodým derivative $\frac{d\nu}{d\nu_\rho}$ is said to exist if it is a quantum random variable; i.e. it takes every x to a bounded operator. If $\frac{d\nu}{d\nu_{\rho_0}}$ exists for some full-rank $\rho_0 \in \mathcal{S}(\mathcal{H})$, then $\frac{d\nu}{d\nu_\rho}$ exists for all full-rank $\rho \in \mathcal{S}(\mathcal{H})$, so there is no need to specify a particular full-rank ρ_0.

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Integrability of a Quantum Random Variable wrt a POVM

Definition

Let $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM such that $\frac{d\nu}{d\nu_\rho}$ exists. A positive quantum random variable $f : X \to \mathcal{B}(\mathcal{H})$ is *ν-integrable* if the function

$$f_s(x) = \mathrm{Tr}\left(s \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} \right)$$

is ν_ρ-integrable for every state $s \in \mathcal{S}(\mathcal{H})$.

If f is ν-integrable then the integral of f with respect to ν, denoted $\int_X f d\nu$, is implicitly defined by the formula

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A particularly Nice Case

If $\nu = \mu I_{\mathcal{H}}$ for a positive complex measure μ then we know that $\frac{d\nu}{d\nu_\rho} = I_{\mathcal{H}}$ and if $f = [f_{i,j}]$ is taken with respect to an orthonormal basis in $\mathcal{H}$ then integration is defined entrywise:

$$\int_X f d\nu = \left[\int_X f_{i,j} d\mu \right].$$

What about Quantum Random Variables that are **not** Positive

Any quantum random variable $f : X \to \mathcal{B}(\mathcal{H})$ can be decomposed as the sum of four positive quantum random variables (e.g. $(\mathrm{Re}f)_+, (\mathrm{Re}f)_-, (\mathrm{Im}f)_+$, and $(\mathrm{Im}f)_-$). The definition of ν-integrable can thus be extended to arbitrary quantum random variables provided all four positive functions are ν-integrable.

Decreasing Rearrangements

One can define continuous majorization in the context of functions in L^1:

Definition

Let $(X, \mathcal{O}(X), \mu)$ be a finite positive measure space and $f \in L^1(X, \mu)$. The *distribution function* of f is $d_f : \mathbb{R} \to [0, \mu(X)]$ defined by

$$d_f(s) = \mu(\{x : f(x) > s\})$$

and the *decreasing rearrangement* of f is $f^\downarrow : [0, \mu(X)] \to \mathbb{R}$ defined by

$$f^\downarrow(t) = \sup\{s : d_f(s) \geq t\}.$$

Majorization

Definition

Let $(X_i, \mathcal{O}(X_i), \mu_i)$, $i = 1, 2$, be finite measure spaces for which $a = \mu_1(X_1) = \mu_2(X_2)$. Then $f \in L^1(X_1, \mu_1)$ is *majorized* by $g \in L^1(X_2, \mu_2)$, denoted $f \prec g$, if

$$\begin{aligned} \int_0^t f^{\downarrow} dx &\leq \int_0^t g^{\downarrow} dx \quad \forall\, 0 \leq t \leq a \\ \text{and } \int_0^a g^{\downarrow} dx &= \int_0^a f^{\downarrow} dx, \end{aligned}$$

where integration is against Lebesgue measure.

Bistochastic Operators

An operator $B : L^1(X_1, \mu_1) \to L^1(X_2, \mu_2)$ between finite measure space where $\mu_1(X_1) = \mu_2(X_2)$ is called *bistochastic, doubly stochastic*, or *Markov*, if

1. B is positive
2. $\displaystyle\int_{X_2} Bf d\mu_2 = \int_{X_1} f d\mu_1$, and
3. $\mathrm{B}1 = 1$

where 1 here refers to the constant function 1 in each of the spaces $L^1(X_i, \mu_i), i = 1, 2$.

Combining results of Hardy-Littlewood-Pólya, Chong, Ryff, and Day

Theorem

Let $(X_i, \mathcal{O}(X_i), \mu_i)$, $i = 1, 2$, be finite measure spaces for which $\mu_1(X_1) = \mu_2(X_2)$. If $f \in L^1(X_1, \mu_1)$ and $g \in L^1(X_2, \mu_2)$ then the following are equivalent:

- $f \prec g$
- $\displaystyle\int_{X_1} \psi(f(x))dx \leq \int_{X_2} \psi(g(x))dx$ *for all convex functions* $\psi : \mathbb{R} \to \mathbb{R}$
- *There is a bistochastic operator B such that $Bg = f$.*

II. The $\mathbf{L}^1$-Norm

The Building Blocks

We wish to find a generalization of the L^1-norm in the POVM context. We require some inequalities...

Lemma

Suppose $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}$ *exists and* $f : X \to \mathcal{B}(\mathcal{H})$ *is* ν*-integrable. Then*

$$\left\| \int_X f(x) d\nu(x) \right\| \leq \int_X \|f(x)\| \left\| \frac{d\nu}{d\nu_\rho}(x) \right\| d\nu_\rho(x).$$

Furthermore, if $\nu = \mu I$ *where* μ *is a positive classical measure on* X *then*

$$\left\| \int_X f(x) d\nu(x) \right\| \leq \int_X \|f(x)\| d\mu(x).$$

For Self-adjoint Quantum Random Variables

Lemma

Suppose $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}$ *exists and* $f : X \to \mathcal{B}(\mathcal{H})$ *is* ν*-integrable and self-adjoint. Then*

$$\left\| \int_X f(x) d\nu(x) \right\| \leq \left\| \int_X \|f(x)\| I_{\mathcal{H}} d\nu(x) \right\|.$$

Candidates for an L^1-Norm

- $\left\| \int_X \|f(x)\| I_{\mathcal{H}} d\nu(x) \right\|$? Reminiscent of the Lebesgue-Bochner norm $\int_X \|f(x)\| d\mu$ on $L^1(X,\mu)\hat{\otimes}_\pi \mathcal{B}(\mathcal{H})$. No. Overestimate in general. Many good functions will not be bounded.
- $\left\| \int_X |f(x)| d\nu(x) \right\|$? No. Does not satisfy the triangle inequality.

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A Worthy L^1-norm

Definition

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ and define

$$\mathcal{L}^1_{\mathcal{H}}(X,\nu) = \mathrm{span}\{f : X \to \mathcal{B}(\mathcal{H}) : \nu\text{-integrable, positive quantum random variable}\}.$$

For every $f \in \mathcal{L}^1_{\mathcal{H}}(X,\nu)$ define

$$\|f\|_1 = \inf\left\{ \left\| \int_X \sum_{k=1}^{4} f_k \, d\nu \right\| : f = f_1 - f_2 + i(f_3 - f_4), f_k \in \mathcal{L}, f_k \geq 0, k = 1, \ldots, 4 \right\}.$$

We may write $\|f\|_{1,\nu}$ to emphasize the POVM ν that f is being integrated against.

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Proposition

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}$ *exists. Then* $\|\cdot\|_1$ *is a semi-norm on* $\mathcal{L}^1_{\mathcal{H}}(X,\nu)$ *such that* $\|f^*\|_1 = \|f\|_1$.

Lemma

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}$ *exists. For all* ν*-integrable quantum random variables* $f : X \to \mathcal{B}(\mathcal{H})$

$$\int_X f d\nu = \int_X \left(\frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} \right) d\nu_\rho I_{\mathcal{H}}$$

and so

$$\|f\|_{1,\nu} \geq \left\| \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} \right\|_{1,\nu_\rho I_{\mathcal{H}}}.$$

Furthermore, this is an equality if $\frac{d\nu}{d\nu_\rho}(x) \in \mathcal{B}(\mathcal{H})^{-1}$.

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$$\int_X |f_s| d\nu_\rho \ \leq \ \|f\|_1.$$

The von Neumann algebra of essentially bounded quantum random variables

Let

$$\begin{aligned} L^\infty_{\mathcal{H}}(X,\nu) &= \{h : X \to \mathcal{B}(\mathcal{H}) \text{ qrv } : \exists M \geq 0, \|h(x)\| \leq M \text{ a.e wrt } \nu\} \\ &= L^\infty(X,\nu_\rho) \,\bar{\otimes}\, \mathcal{B}(\mathcal{H}) \end{aligned}$$

Note that the norm this comes with is defined as

$$\|f(x)\|_\infty := \Big\| \|f(x)\| \Big\|_{L^\infty(X,\nu_\rho)}$$

since $\|f(x)\| \in L^\infty(X,\nu_\rho)$.

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Proposition

Suppose $\mathcal{H} = \mathbb{C}^n$, $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho} \in M_n$ *is invertible almost everywhere* ($\frac{d\nu}{d\nu_\rho} \in M_n^{-1}$ *a.e.*), *and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X, \nu)$. *For* $f \in \mathcal{L}^1_{\mathcal{H}}(X, \nu)$ *self-adjoint we have*

$$\|f\|_1 \leq \left\| \int_X |f(x)| d\nu \right\| \leq \left\| \int_X \|f(x)\| I_n d\nu \right\| \leq n \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \left\| \frac{d\nu}{d\nu_\rho}^{-1} \right\|_\infty \|f\|_1.$$

Recall for $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ we have

$$\mathcal{L}^1_{\mathcal{H}}(X,\nu) = \mathrm{span}\{f : X \to \mathcal{B}(\mathcal{H}) : \nu\text{-integrable, positive quantum random variable}\}.$$

Define $\mathcal{I} = \{f \in \mathcal{L}^1_{\mathcal{H}}(X,\nu) : \|f\|_1 = 0\}$ and let $L^1_{\mathcal{H}}(X,\nu) = \mathcal{L}^1_{\mathcal{H}}(X,\nu)/\mathcal{I}$. The previous lemma implies that the 1-topology on $L^1_{\mathcal{H}}(X,\nu)$ is stronger than the topology $(f_n)_s \to f_s$ for all $s \in \mathcal{S}(\mathcal{H})$.

Theorem

$L^1_{\mathcal{H}}(X,\nu)$ is a Banach space, that is, it is complete in the 1-norm for $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ where $\frac{d\nu}{d\nu_\rho}$ exists.

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How to Relate $L^\infty_{\mathcal{H}}(X,\nu)$ and $L^1_{\mathcal{H}}(X,\nu)$

Proposition

Suppose $\frac{d\nu}{d\nu_\rho}(x) \in \mathcal{B}(\mathcal{H})^{-1}$ *for all* $x \in X$ *and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X,\nu)$*. There is a natural inclusion of* $L^\infty_{\mathcal{H}}(X,\nu)$ *in* $L^1_{\mathcal{H}}(X,\nu)$ *with*

$$\|g\|_1 \leq 2\|g\|_\infty \|\nu(X)\|, \quad \forall g \in L^\infty_{\mathcal{H}}(X,\nu).$$

Moreover, $L^\infty_{\mathcal{H}}(X,\nu)$ *is dense in* $L^1_{\mathcal{H}}(X,\nu)$ *in the state topology,* $(f_n)_s \to f_s$ *for all* $s \in \mathcal{S}(\mathcal{H})$*.*

Finite vs Infinite Dimensions

This proposition implies that if $\mathcal{H} = \mathbb{C}^n$ then $L^1_{\mathcal{H}}(X, \nu) = \overline{L^\infty_{\mathcal{H}}(X, \nu)}^{\|\cdot\|_1}$. In infinite dimensions this will not be the case: consider $X = [0, 1]$, $\mathcal{H}$ countably infinite dimensional, and $\nu = \mu I_{\mathcal{H}}$ where μ is Lebesgue measure. Then $f(x) = \sum_{n \geq 1} 2^n \chi_{(\frac{1}{2^n}, \frac{1}{2^{n-1}})}(x) e_{n,n}$ cannot be approximated by essentially bounded functions in the 1-norm.

III. Bounded Multipliers

A Natural Pairing

Although $L^{\infty}_{\mathcal{H}}(X,\nu)$ is not the dual space of $L^{1}_{\mathcal{H}}(X,\nu)$, we can think of it as a generalization of the dual space. Consider the following "natural pairing" or "bracket"

$$\langle\cdot,\cdot\rangle : L^{1}_{\mathcal{H}}(X,\nu) \times L^{\infty}_{\mathcal{H}}(X,\nu) \to \mathcal{B}(\mathcal{H})$$

given by

$$\langle f,g\rangle = \int_X fg \; d\nu.$$

The main trouble with this is that fg may fail to be in $L^{1}_{\mathcal{H}}(X,\nu)$, or to put it another way, multiplication by $g \in L^{\infty}_{\mathcal{H}}(X,\nu)$ could be an unbounded operator on $L^{1}_{\mathcal{H}}(X,\nu)$.

Cauchy-Schwarz

Lemma

For all $f \in L^1_{\mathcal{H}}(X,\nu)$ and $g \in L^\infty(X,\nu_\rho)$ one has

$$\|f \cdot gI_{\mathcal{H}}\|_1 = \|gI_{\mathcal{H}} \cdot f\|_1 \leq 2\|f\|_1\|g\|_\infty.$$

Lemma

If $f \in L^1_{\mathcal{H}}(X,\nu)$ and $g \in L^\infty(X,\nu_\rho)$ then

$$\|\langle f, gI_{\mathcal{H}}\rangle\| \leq 4\|f\|_1\|g\|_\infty.$$

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Proposition

Suppose $\nu = \mu I_{\mathcal{H}} \in \mathrm{POVM}_{\mathcal{H}}(X)$ *where* μ *is a positive, finite measure on* X*. If* $f \in L^1_{\mathcal{H}}(X,\nu)$ *and* $A \in \mathcal{B}(\mathcal{H})$ *then* Af *and* fA *are in* $L^1_{\mathcal{H}}(X,\nu)$ *with*

$$\|Af\|_1 = \|fA\|_1 \leq 4\left(1 + \|A\|^2\right)\|f\|_1.$$

Corollary

Suppose $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}(x) \in \mathcal{B}(\mathcal{H})^{-1}$ *for all* $x \in X$ *and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X,\nu)$*. If* $f \in L^1_{\mathcal{H}}(X,\nu)$ *and* $A \in \mathcal{B}(\mathcal{H})$ *then*

$$\left\| \frac{d\nu}{d\nu_\rho}^{-1/2} A \frac{d\nu}{d\nu_\rho}^{1/2} f \right\|_1 = \left\| f \frac{d\nu}{d\nu_\rho}^{1/2} A \frac{d\nu}{d\nu_\rho}^{-1/2} \right\|_1 \leq 4\left(1 + \|A\|^2\right)\|f\|_1.$$

Proposition

Suppose $\nu = \mu I_{\mathcal{H}} \in \mathrm{POVM}_{\mathcal{H}}(X)$ *where* μ *is a positive, finite measure on* X. *If* $f \in L^1_{\mathcal{H}}(X, \nu)$ *and* $A \in \mathcal{B}(\mathcal{H})$ *then* Af *and* fA *are in* $L^1_{\mathcal{H}}(X, \nu)$ *with*

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In finite dimensions, every multiplication operator is bounded, assuming some conditions on the Radon-Nikodým derivative.

Proposition

Suppose $\mathcal{H} = \mathbb{C}^n$, $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho}(x) \in M_n^{-1}$ *for all* $x \in X$ *and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X, \nu)$. *If* $f \in L^1_{\mathcal{H}}(X, \nu)$ *and* $g \in L^\infty_{\mathcal{H}}(X, \nu)$ *then* $fg \in L^1_{\mathcal{H}}(X, \nu)$ *with*

$$\|fg\|_1 \leq n \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \left\| \frac{d\nu}{d\nu_\rho}^{-1} \right\|_\infty \|f\|_1 \|g\|_\infty.$$

However, if the boundedness condition is dropped, multipliers can become unbounded even in finite dimensions.

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However, if the boundedness condition is dropped, multipliers can become unbounded even in finite dimensions.

The set of bounded multipliers is difficult to characterize. We therefore restrict our consideration to only those arising from $L^\infty(X, \nu_\rho)$:
Define the following subspace of linear functionals on $L^1_{\mathcal{H}}(X, \nu)$:

$$\mathcal{F}(X, \nu) = \text{span}\{\text{Tr}(s\langle \cdot, gI_{\mathcal{H}}\rangle) : s \in \mathcal{S}(\mathcal{H}), g \in L^\infty(X, \nu_\rho)\}.$$

Proposition

The family $\{\langle \cdot, gI_{\mathcal{H}}\rangle : g \in L^\infty(X, \nu_\rho)\}$ is separating and $\mathcal{F}(X, \nu)$ is a separating subspace of linear functionals on $L^1_{\mathcal{H}}(X, \nu)$.

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Positivity Detection

Unlikely that this family recovers the 1-norm on $L^1_{\mathcal{H}}(X,\nu)$.

However, we can show that it detects positivity:

Lemma

Suppose $f \in L^1_{\mathcal{H}}(X,\nu)$, then $f \geq 0$ if and only if $\langle f, gI_{\mathcal{H}}\rangle \geq 0$ for all $g \in L^\infty(X,\nu_\rho)$ such that $g \geq 0$.

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Weak Convergence

Definition

We say a sequence $\{f_i\}_{i\geq 1}$ in $L^1_{\mathcal{H}}(X,\nu)$ is *weakly converging* to $f \in L^1_{\mathcal{H}}(X,\nu)$ if it is converging weakly with respect to the family $\mathcal{F}(X,\nu)$. This is the same as

$$\langle f_i, gI_{\mathcal{H}}\rangle \to \langle f, gI_{\mathcal{H}}\rangle, \quad \forall g \in L^\infty(X,\nu_\rho)$$

with convergence in the ultraweak topology of $\mathcal{B}(\mathcal{H})$.

IV. Bistochastic Operators

Bistochastic Operators

Definition

A linear operator B is called a *bistochastic operator* on $L^1_{\mathcal{H}}(X,\nu)$ if

1. B is positive,
2. $\int_X Bfd\nu = \int_X fd\nu, \quad \forall f \in L^1_{\mathcal{H}}(X,\nu)$,
3. $BI_{\mathcal{H}} = I_{\mathcal{H}}$,

where $I_{\mathcal{H}}$ above refers to the constant function $I_{\mathcal{H}}$ in $L^1_{\mathcal{H}}(X,\nu)$. The set of all bistochastic operators on $L^1_{\mathcal{H}}(X,\nu)$ is denoted by $\mathfrak{B}(X,\nu)$.

Lemma

Every bistochastic operator B is self-adjoint, meaning that for every $f \in L^1_{\mathcal{H}}(X, \nu)$ we have $B(f^*) = B(f)^*$.

Lemma

Every bistochastic operator takes $L^\infty_{\mathcal{H}}(X, \nu)$ to itself and is bounded in the ∞-norm. Furthermore, it is contractive on all self-adjoint functions.

Proposition

Every bistochastic operator is contractive with respect to the $\| \cdot \|_1$-norm.

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Proposition

Every bistochastic operator is contractive with respect to the $\|\cdot\|_1$-norm.

The set of bistochastic operators on the classical $L^1(X,\mu)$ is denoted $\mathfrak{B}(L^1(X,\mu))$.

Theorem

If $\nu = \mu I_{\mathcal{H}}$ for some finite, positive measure μ, then every $B \in \mathfrak{B}(L^1(X,\mu))$ extends to a bistochastic operator in $\mathfrak{B}(X,\nu)$ by the formula

$$B\,(fA) = B(f)A, \quad \forall f \in L^1(X,\mu),\ A \in \mathcal{B}(\mathcal{H}).$$

We will refer to the extension developed in the above theorem by B as well and the set of such bistochastic operators as $\mathfrak{B}(L^1(X,\mu))$ still. We have no example of a bistochastic operator on $L^1_{\mathcal{H}}(X,\mu I_{\mathcal{H}})$ that does not arise in this way.

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Corollary

If $\nu = \mu I_{\mathcal{H}}$ *for some finite, positive measure* μ*, then for every* $B \in \mathfrak{B}(L^1(X,\mu))$*,* $f \in L^1(X,\mu), A \in \mathcal{B}(\mathcal{H})$ *and* $g \in L^\infty(X,\mu)$

$$\langle B(fA), gI_{\mathcal{H}} \rangle = \langle B(f), g \rangle A.$$

Lastly, we turn to topology again. Suppose $B_i, B \in \mathfrak{B}(X, \nu), i \geq 1$. We say that B_i is *WOT-convergent* to B if $B_i f$ weakly converges to Bf for all $f \in L^1_{\mathcal{H}}(X, \nu)$, that is

$$\langle B_i(f), gI_{\mathcal{H}} \rangle \to \langle B(f), gI_{\mathcal{H}} \rangle, \quad \forall f \in L^1_{\mathcal{H}}(X, \nu), g \in L^\infty(X, \nu_\rho)$$

in the ultraweak topology on $\mathcal{B}(\mathcal{H})$.

V. Majorization of Quantum Random Variables

Variants of Multivariate Majorization

Recall that if $f \in L^1_{\mathcal{H}}(X, \mu I)$ and $s \in \mathcal{T}(\mathcal{H})$ then we define $f_s \in L^1(X, \mu)$ by

$$f_s(x) = \mathrm{Tr}(sf(x)) \in L^1(X, \mu).$$

We now introduce several possible majorization partial orders which relate to multivariate majorization

Definition

Suppose $f, g \in L^1_{\mathcal{H}}(X, \mu I)$ and are self-adjoint where μ is a finite, positive, complex measure. We say that

1. $f \prec g$ if there exists a bistochastic operator $B \in \mathfrak{B}(L^1(X, \mu))$ such that $Bg = f$,
2. $f \prec_T g$ if $f_t \prec g_t$ for all $t \in \mathcal{T}(\mathcal{H})_{sa}$, and
3. $f \prec_S g$ if $f_s \prec g_s$ for all $s \in \mathcal{S}(\mathcal{H})$.

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Relating the Three Partial Orders

Proposition

For $f, g \in L^1_{\mathcal{H}}([0,1], \mu I)$ *self-adjoint we have that*

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A Result of Komiya

Komiya (1983): For $X, Y \in M_{m,n}(\mathbb{C})$, we have that $X \prec Y$ if and only if $\psi(X) \leq \psi(Y)$ for every real-valued, permutation-invariant, convex function ψ on $M_{m,n}(\mathbb{C})$.

(Note: The convex hull of the permutation matrices is the set of bistochastic matrices.)

We use the notation C_ϕ to denote the right-composition operator: $C_\phi(f) = f \circ \phi$, and $\mathcal{P}_{\text{inv}}$ to denote the set of all invertible measure-preserving maps of X, where the measure is understood by context. If $\phi \in \mathcal{P}_{\text{inv}}$ then C_ϕ is a bistochastic operator.

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Brown (1966) proved a similar convexity result for bistochastic operators on L^1 under some conditions on the measure space. Namely, the convex hull $\mathrm{conv}(C_\phi : \phi \in \mathcal{P}_{\mathrm{inv}})$ of the composition operators of invertible measure-preserving maps is dense in the bistochastic operators in the weak operator topology arising from L^p for every $1 < p < \infty$.

Proposition

Suppose X is a product of unit intervals and μ is the corresponding product of Lebesgue measures. If B is a bistochastic operator in $\mathfrak{B}(L^1(X,\mu))$ then there exists a sequence of bistochastic operators $B_i \in \mathrm{conv}(C_\phi : \phi \in \mathcal{P}_{\mathrm{inv}})$ such that B_i is WOT-convergent to B. Moreover, $\mathfrak{B}(L^1(X,\mu))$ is WOT-compact and convex.

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Definition

A real-valued convex function $\psi : L^1_{\mathcal{H}}(X, \mu I) \to \mathbb{R}$ is said to be *permutation-invariant* if for every $\sigma \in \mathcal{P}_{\text{inv}}$ we have

$$\psi(f \circ \sigma) = \psi(f) \quad \forall f \in L^1_{\mathcal{H}}(X, \mu I).$$

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