Amenability, proximality and higher order syndeticity

Matthew Kennedy
joint work with Sven Raum and Guy Salomon

University of Waterloo, Waterloo, Canada

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Proximality and strong proximality

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There is a unique universal minimal proximal flow $\partial_p G$ and a unique universal minimal strongly proximal flow $\partial_{sp} G$. For every minimal proximal flow X there is a surjective G-map $\partial_p G \to X$. Similarly for $\partial_{sp} G$.

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Statements about specific flows translate to statements about universal flows. E.g. G has a free proximal flow iff $\partial_p G$ is free.

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Proof.

For a minimal G-flow X, choose an irreducible affine G-subflow $K \subseteq \mathrm{Prob}(X)$. Then $\overline{\partial K}$ is a strongly proximal G-flow and $\overline{\partial K}$ is trivial iff K is trivial iff there is an invariant measure in $\mathrm{Prob}(X)$. $\square$

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Example (Connected groups)

For locally compact G with $P \leq G$ amenable and co-compact,

$$\partial_{sp} G = G/Q \quad \text{for} \quad \text{closed } Q \geq P.$$

For G connected semi-simple real Lie group with finite center and no compact factors,

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Example (Discrete groups)

For non-amenable discrete G, $\partial_{sp} G$ is non-trivial and extremally disconnected (Stone space!). Hence non-metrizable.

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For discrete G, the reduced C-algebra $\mathrm{C}^*_\lambda G$ is simple iff G has a topologically free strongly proximal flow.*

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Theorem (K 2018)

The reduced C-algebra $\mathrm{C}^*_\lambda G$ is simple iff G has no amenable confined subgroups.*

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For $x \in X$, there is $h_i \in C_G(g)$ with $\lim h_i x = \lim h_i g x = x$. Then $gx = \lim g h_i x = \lim h_i g x = x$, contradicting faithfulness. □

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Key point: If $\partial_p G$ is non-trivial then it is free. This is not true for $\partial_{sp} G$. (Reminiscent of the fact that LG has a unique trace iff LG is a factor, but $C^*_\lambda G$ can have a unique trace without being simple.)

Injectivity

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Goal: Identify these Boolean algebras, thereby giving "concrete" descriptions of $\partial_{p} G$ and $\partial_{sp} G$.

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Theorem (KRS 2020)

The universal minimal (strongly) proximal flow is isomorphic to the spectrum of any invariant Boolean algebra of (strongly) completely syndetic subsets of G.

Characterizations of complete syndeticity

Proposition (KRS 2020)

Let X be a proximal flow. For open $U \subseteq X$ and $x \in X$, the return set $U_x = \{g \in G : gx \in U\}$ is completely syndetic.

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The following are equivalent for $A \subseteq G$:

1. The set A is completely syndetic, i.e. for each n, there is finite $F \subseteq G$ such that $FA^n = G^n$.
2. For every n and every mean m on G^n that is invariant under left translation by G, $m(A^n) > 0$
3. The closur $\overline{A} \subseteq \beta G$ contains a closed right ideal of βG.

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Similar characterizations hold for strongly completely syndetic subsets.

Examples of completely syndetic subsets

Example

A subset $A \subseteq \mathbb{Z}$ is syndetic if and only if it has "bounded gaps," meaning there is $k \in \mathbb{N}$ such that for all $a \in \mathbb{Z}$,

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A subset $A \subseteq Z$ is completely syndetic if and only if for every n, A^n has "bounded diagonal gaps," meaning there is $k \in \mathbb{N}$ such that for any $(a_1, \ldots, a_n) \in \mathbb{Z}^n$,

$$\{(a_1, \ldots, a_n), (a_1+1, \ldots, a_n+1), \ldots, (a_1+k, \ldots, a_n+k)\} \cap A^n \neq \emptyset.$$

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Fact: The group $\mathbb{Z}$ does not contain disjoint completely syndetic subsets.

Example 1 (the integers)

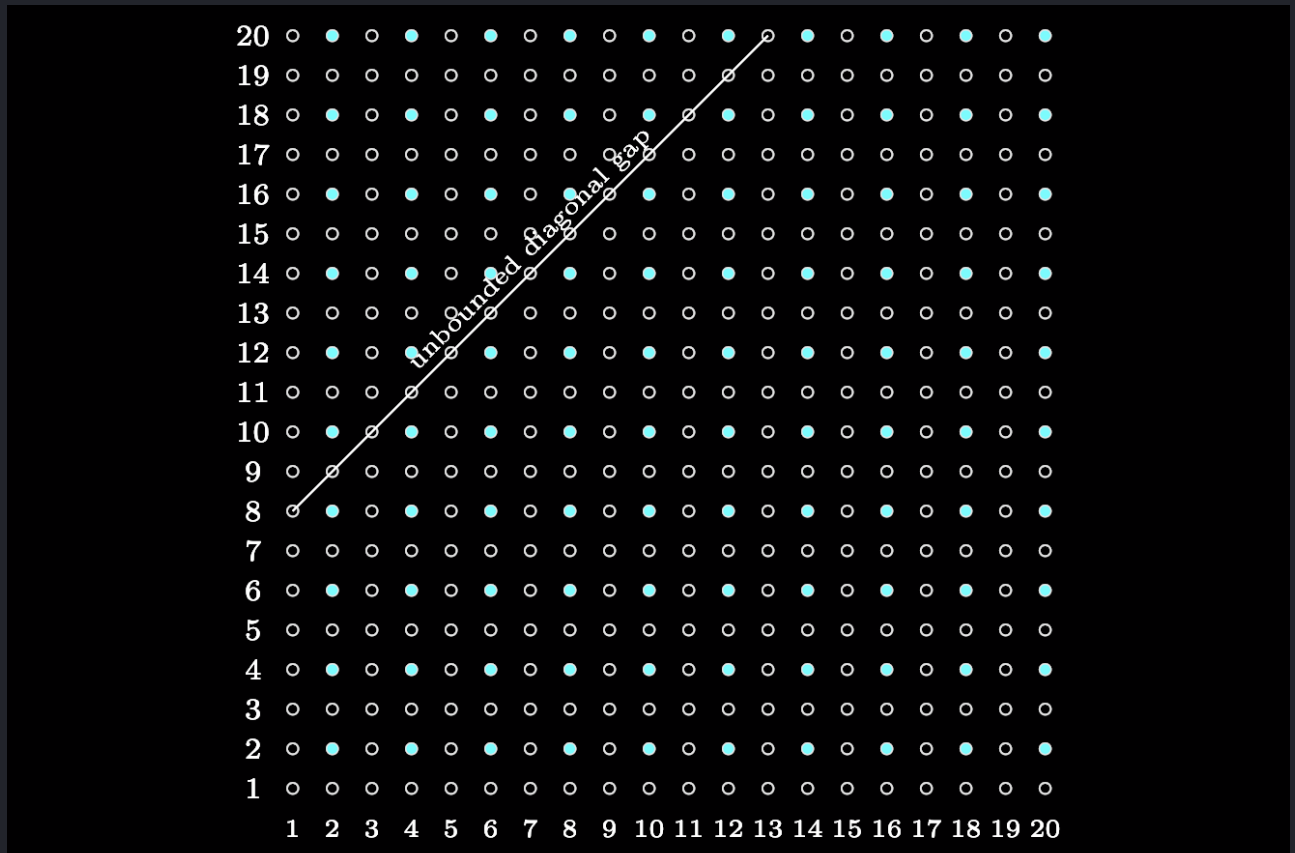

FIGURE 1. The subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$ is syndetic but not 2-syndetic since there are arbitrarily long diagonal segments in $\mathbb{Z} \times \mathbb{Z}$ that do not intersect $2\mathbb{Z} \times 2\mathbb{Z}$.

Example 2 (the integers)

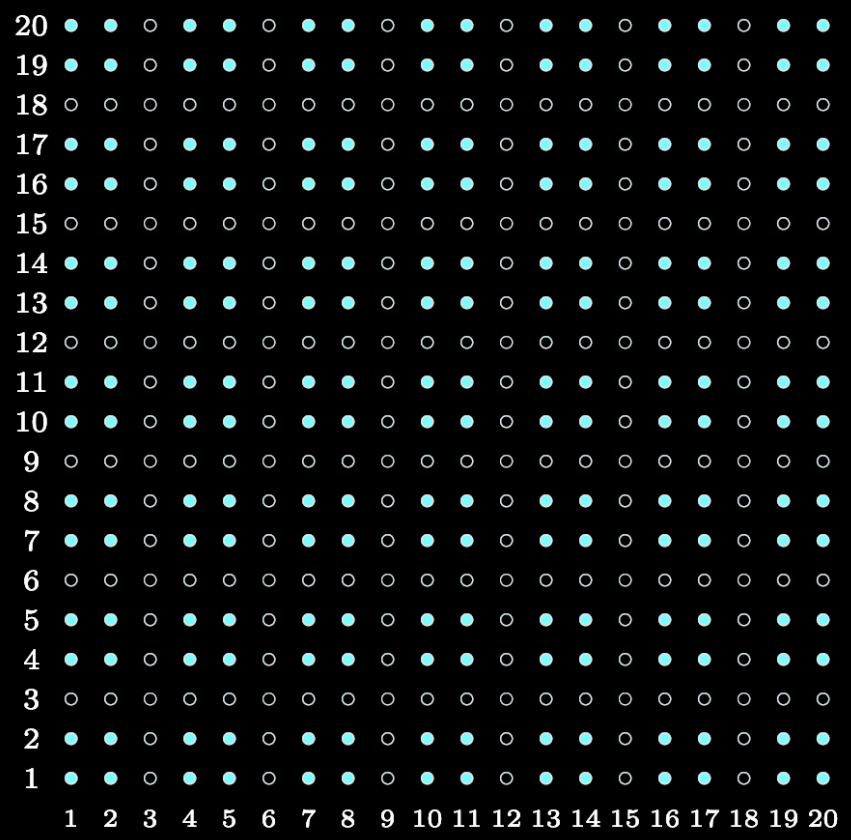

FIGURE 2. The subset $\mathbb{Z} \setminus 3\mathbb{Z} \subseteq \mathbb{Z}$ is 2-syndetic but not 3-syndetic since for $k \in \mathbb{N}$, every element in the set $\{(1,2,3),(2,3,4),(4,5,6),\ldots,(1+k,2+k,3+k)\}$ has an entry that is a multiple of 3, implying that the set does not intersect A^3.

Example 3 (the integers)

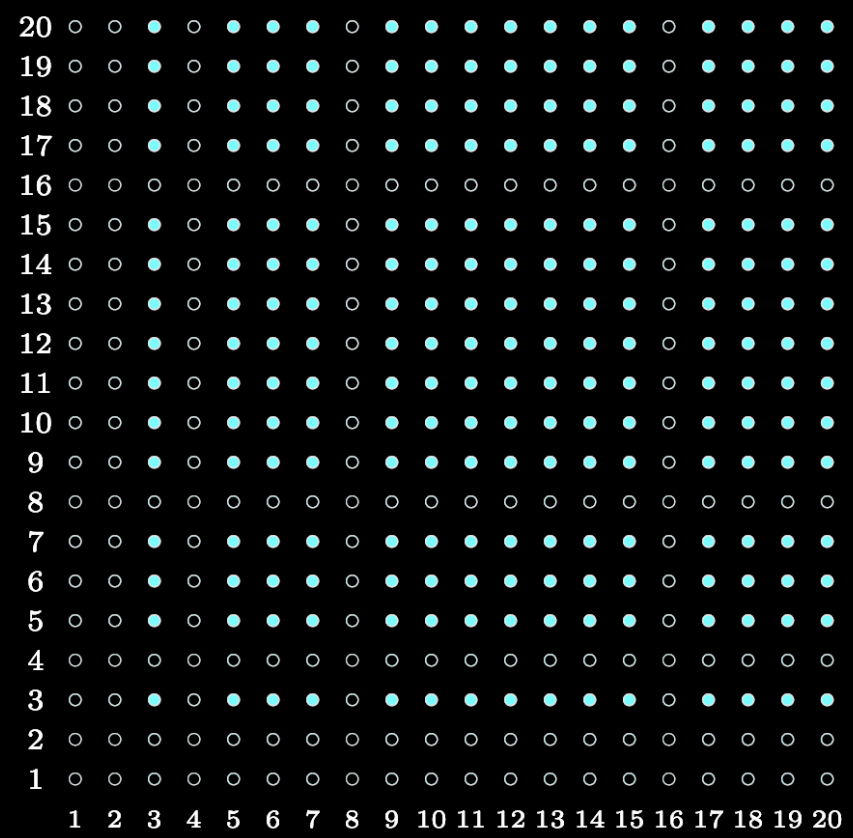

FIGURE 3. The complement of the set of powers of 2 in $\mathbb{Z}$ is completely syndetic, and in particular is 2-syndetic.

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Consider the free group $\mathbb{F}_2 = \langle a, b \rangle$. For $w \in \mathbb{F}_2$, let

$$B_w = \{g \in G : g = wg' \text{ in reduced form}\}.$$

Can show by hand that B_a and B_b are strongly completely syndetic. Alternatively, B_a is the return set $B_a = U_{a^\infty}$ where U is the set of infinite reduced words beginning with a in the hyperbolic boundary $\partial\mathbb{F}_2$.

Characterization of (strong) amenability

Theorem (KRS 2020)

A discrete group G is not strongly amenable if and only if there is a proper normal subgroup $H \leq G$ such that for every finite subset $F \subseteq G \setminus H$, there is a completely syndetic subset $A \subseteq G$ satisfying $FA \cap A = \emptyset$.

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Upshot: Measures the difference between strong amenability and amenability.

“Coloring” criterion for strong amenability

For finite $F \subseteq G$, an F-**coloring** of G is a pair (K, k) consisting of finite colors $K \subseteq G$ and $k : 2_2^G \to K$ such that

$$Fk(g_1, g_2)\{g_1, g_2\} \cap k(h_1, h_2)\{h_1, h_2\} = \emptyset \quad \text{for all} \quad g_1, g_2, h_1, h_2 \in G.$$

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"Coloring" criterion for strong amenability

For finite $F \subseteq G$, an F-**coloring** of G is a pair (K, k) consisting of finite colors $K \subseteq G$ and $k : 2_2^G \to K$ such that

$$Fk(g_1, g_2)\{g_1, g_2\} \cap k(h_1, h_2)\{h_1, h_2\} = \emptyset \quad \text{for all} \quad g_1, g_2, h_1, h_2 \in G.$$

Here 2_2^G denotes the the subsets of G of size 2.

Theorem (KRS 2020)

The group G is not strongly amenable if and only if for every finite $F \subseteq G$, G has an F-coloring.

Observation: Proof of FTV can be translated to give a construction of F-colorings.

Symmetric higher order syndeticity

Definition

A subset $A \subseteq G$ is **symmetrically (completely, strongly completely) syndetic** if for any finite subsets $F_1, F_2 \subseteq G$, the set

$$(\cap_{f_1 \in F_1} f_1 A) \cap (\cap_{f_2 \in F_2} f_2 A^c)$$

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Theorem

A subset $A \subseteq G$ is symmetrically (completely, strongly completely) syndetic iff for every pair of finite subsets $F_1 \subseteq A$ and $F_2 \subseteq A^c$, the set

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is (completely, strongly completely) syndetic.

Second characterization of (strong) amenability

Theorem (KRS 2020)

A discrete group G is not strongly amenable if and only if it contains a symetrically completely syndetic subset.

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Problem (Glasner-Tsankov-Weiss-Zucker)

Characterize dense orbit sets.

Theorem (KRS 2020)

A subset $A \subseteq G$ is a dense orbit set iff A^c does not contain a symmetrically syndetic subset.

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Problem

Is there a direct proof that G is Choquet-Deny iff G is strongly amenable?

Thanks!