Cu-nuclearity implies LLP and exactness

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May, 2021

Outline

- Motivation

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- The main result

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- Some ideas of the proof

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Note: any amenable group is K-amenable but the converse is NOT true in general. The converse is not true: example $G = \mathbb{F}_2$

Skandalis's K-nuclearity, 1988

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- If A is nuclear then A is K-amenable but the converse is not necessarily true.
- Example $C^*(\mathbb{F}_2)$

Cuntz nuclear and weakly Cuntz nuclear

Def

A is **Cu-nuclear** if the canonical quotient map

$$\pi : A \otimes_{max} B \to A \otimes_{min} B$$

induces an isomorphism

$$Cu(A \otimes_{max} B) \cong Cu(A \otimes_{min} B)$$

for all C*-alg B.

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A is **weakly Cu-nuclear** if

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weakly Cu-nuclear $\Rightarrow$ **nuclear** ?

Main result

Theorem (I. Kučerovský)

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Question:

If A is simple and $A \otimes_{min} B = A \otimes_{max} B$ for all simple B then is A nuclear?

Main result

Corollary

If a C*-algebra with finitely many ideals is weakly Cu-nuclear then it is exact and has the LLP.

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If a C*-algebra is Cu-nuclear then it is nuclear.

Corollary

If G is a C*-simple group and if $Cu(C^*(G)) \cong Cu(C^*_r(G))$
$\Longrightarrow$ G is amenable.

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- $\alpha(a^* a) = \alpha(a)^2, \forall a \in A \odot B$, called C*-identity
- Fact: Such a norm must be a **cross-norm**: $\alpha(a \odot b) = ||a||||b||, \forall a \odot b \in A \odot B$
- All C*-norms agree on elementary tensors but not necessarily on sums of elementary tensors!

Minimal tensor product

Minimal tensor product $A \otimes_{min} B$

Let $\phi_1 : A \to B(H_1)$ and $\phi_2 : B \to B(H_2)$ be faithful rep. then $\phi : A \odot B \to B(H_1 \otimes H_2)$ is a faithful rep. Set

$$||x||_{min} = ||\phi(x)||$$

Facts: $||x||_{min}$ is a C*-norm ind. of ϕ_1, ϕ_2 and smaller than any other C*-norm.

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$$A \otimes_{min} B = \overline{A \odot B}^{|| \, ||_{min}}$$

Minimal (i.e. spatial) tensor product vs Maximal tensor product

Maximal tensor product

Define the maximal norm by

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Note

- If $||x||_{min} = ||x||_{max}, \forall x \in A \odot B$ for any C*-alg B then A is nuclear.

Nuclear vs exact C*-algebras

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A C*-alg. A is **nuclear** if

for all C*-alg B

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Exact C*-alg.

A C*-alg. A is **exact** if whenever

$$0 \to J \to B \to B/J \to 0$$

is exact, then

$$0 \to J \otimes_{min} A \to B \otimes_{min} A \to (B/J) \otimes_{min} A \to 0$$

is exact.

Nuclear vs exact

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Nuclear $\Rightarrow$ exact because

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Note

Choi algebra $C_r^*(\mathbb{F}_2 * \mathbb{F}_3)$ is exact but not nuclear.

The Cuntz semigroup

Cuntz equivalence

If $a, b \in (A \otimes \mathbb{K})_+$ then $a \preceq b$ if there exists $r_n \in A \otimes \mathbb{K}$ such that

$$a = \lim_{n \to \infty} r_n b r_n^*$$

$$a \sim b$$

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Cuntz semigroup

Define the Cuntz semigroup as

$$Cu(A) = (A \otimes \mathbb{K})_+/\sim$$

The Cuntz semigroup and ideals

If I is an ideal in a C*-alg C then

$$\{[a] \in Cu(C) | a \in (I \otimes \mathbb{K})_+\}$$

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Conversely: if J is an order ideal in $Cu(C)$ then the set

$$\{c \in C | [cc^*] \in J\}$$

is an ideal in C.

Steps in the proof

there is a canonical surjective map

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$$\ker h = I$$

with $I \subset A \otimes_{max} B$

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- However, if $I = \ker h$ is non-trivial, then we get one extra ordered ideal in $Cu(A \otimes_{max} B)$ vs. $Cu(A \otimes_{min} B)$
- Hence $A \otimes_{max} B \to A \otimes_{min} B$ is an isomorphism for all simple B or more generally, if A and B have finitely many ideals.

Applications

Local Lifting Property LLP

- A has LLP if every u.c.p map from A to B/J is locally liftable to B. Here J is any two-sided ideal in B. "locally" means $\forall E$ finite-dim. subspace in A there is a "lift" complete contraction from E to B.

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- Kirchberg: A has the LLP $\Longleftrightarrow A \otimes_{max} B(H) = A \otimes_{min} B(H)$.
- Use $B(H)$ as a test algebra in main result to obtain LLP.

Exactness of the algebra

Finitely many ideals + Weakly Cu implies exactness

- Using the Calkin algebra as a test algebra we have

$$A \otimes_{min} (B(H)/K(H)) = A \otimes_{max} (B(H)/K(H))$$

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- which is an equivalent definition for exactness for A.

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- Cu-nuclear (without finitely many ideals) implies nuclearity

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- (**Ciuperca, Roberts, Santiago**)

$$0 \to Cu(\ker h) \to Cu(A \otimes_{max} B) \to Cu(A \otimes_{min} B) \to 0$$

- Then $\ker h = 0$ so we deduce that h is an isomorphism

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- If G is *weakly Cu-nuclear* then $Cu(C^*(G)) \cong Cu(C_r^*(G))$
- Then $C^*(G)$ is simple, i.e. G is amenable.