

Graph operator systems generated by a projection

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Graph operator systems

Let Γ be a connected graph with vertex set $\{1, 2, \dots, n\}$.

Graph operator system of Γ :

$$\mathcal{S}(\Gamma) = \text{span} \{e_{ij} \mid i=j \text{ or } ij \in E(\Gamma)\}$$

$$\uparrow = \{X \in M_n(\mathbb{C}) \mid x_{ij} = 0 \Rightarrow \begin{matrix} i \neq j \\ \& ij \notin E(\Gamma) \end{matrix}\}$$

D_n -bimodule

$$\mathcal{R}(\Gamma) \subseteq \mathcal{S}(\Gamma)$$

$$\mathcal{R}(\Gamma) = \{X \in \mathcal{S}(\Gamma) \mid ij \in E(\Gamma) \Rightarrow x_{ij} \neq 0\}$$

$$\underline{X \in \mathcal{R}(\Gamma)} \iff \langle X, D_n \rangle = \mathcal{S}(\Gamma), \text{ as a } D_n\text{-bimodule.}$$



$$\mathcal{S}(\Gamma) = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}$$

$$\mathcal{R}(\Gamma) = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}$$

\uparrow

Question

For which graphs Γ does $\mathcal{R}(\Gamma)$ contain a projection?

IFPG

Some simple reformulations

Proposition

TFAE:

1. $\mathcal{R}(\Gamma)$ contains a projection;
2. $\mathcal{R}(\Gamma)$ contains a Hermitian unitary;
3. $\mathcal{R}(\Gamma)$ contains a normal matrix with at most two distinct eigenvalues.

$$X \in \mathcal{R}(\Gamma) \implies \underbrace{\alpha X + \beta I}_{\substack{\alpha \neq 0 \\ \alpha \in \mathbb{C} \quad \beta \in \mathbb{C}}} \in \mathcal{R}(\Gamma)$$

$$P \mapsto 2P - I$$

Paths

$$P_n = \textcircled{1} - \textcircled{2} - \dots - \textcircled{n}$$

$$\mathcal{R}(P_n) = \begin{bmatrix} * & * & & & 0 \\ * & * & * & & \\ * & * & * & * & \\ 0 & \ddots & \ddots & \ddots & \end{bmatrix}$$

Proposition (folklore)

If $X \in \mathcal{R}(P_n)$ is normal, then X has n distinct eigenvalues.

In particular, if $n \geq 3$, then $\mathcal{R}(P_n)$ is projectionless.

pf $X \in \mathcal{R}(P_n)$

$\Rightarrow I, X, X^2, \dots, X^{n-1}$ are lin indep

$$\begin{pmatrix} * & * & \textcircled{0} & \dots \\ * & * & \textcircled{*} & \dots \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}, \begin{pmatrix} * & * & \textcircled{*} & \dots \\ * & \ddots & \ddots & \ddots \\ * & \ddots & \ddots & \ddots \end{pmatrix}$$

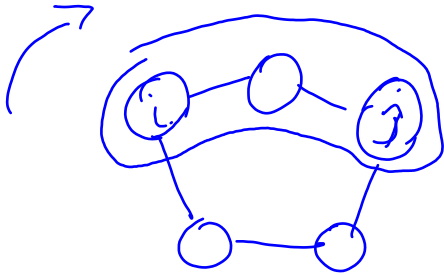
\Rightarrow min poly of X has degree n ,

$\Rightarrow X$ has n distinct eigenvalues. \square

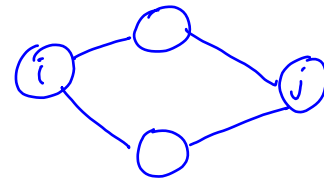
Unique shortest path obstruction

Proposition (Fallat et al. 2013)

If $\text{dist}_\Gamma(i, j) \geq 2$ and this is attained by a unique path in Γ , then $\mathcal{R}(\Gamma)$ is projectionless.



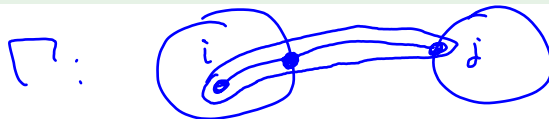
$\mathcal{R}(C_5) \not\subseteq \text{projection}$



$\mathcal{R}(C_4) \ni \text{projection}$

Example

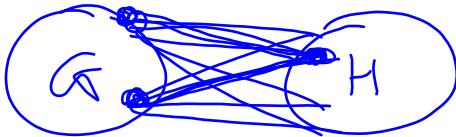
If Γ has a cut edge and at least 3 vertices, then $\mathcal{R}(\Gamma)$ is projectionless.



The join of two graphs

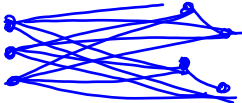
Let G, H be any graphs (possibly disconnected).

The join $G \vee H$ is connected and usually lacks the unique shortest path obstruction.



$$\mathcal{R}(G \vee H) = \left[\begin{array}{c|c} \mathcal{R}(G) & \begin{array}{c} \text{no 0's} \\ \downarrow \\ * \end{array} \\ \hline * & \mathcal{R}(H) \end{array} \right]$$

eg / $3K_1 \vee 2K_2$



$$\mathcal{R}(3K_1 \vee 2K_2) = \left[\begin{array}{c|c} * & * \\ * & * \\ * & * \\ \hline * & * * * \\ * & * * * \\ * & * * * \end{array} \right]$$

Theorem (Monfared-Shader 2016)

If G and H are connected graphs with $|G| = |H|$, then $\mathcal{R}(G \vee H)$ contains a projection.

Necessary condition (connected G, H)

Write $\sigma'(X) = \sigma(X) \setminus \{0, 1\}$.

Proposition (L-Oblak-Šmigoc 2020)

If G and H are connected and $\mathcal{R}(G \vee H)$ contains a projection, then there exist positive semidefinite contractions $A \in \mathcal{R}(G)$ and $C \in \mathcal{R}(H)$ so that $\emptyset \neq \sigma'(A) = \sigma'(C)$.

Necessary condition (arbitrary G, H)

Theorem (L-Oblak-Šmigoc 2020)

If $G = \bigcup G_i$ and $H = \bigcup H_j$ where G_i, H_j are connected, and $\mathcal{R}(G \vee H)$ contains a projection, then there exist positive semidefinite contractions $A \in \mathcal{R}(G)$ and $C \in \mathcal{R}(H)$ so that

- 1. $\sigma'(A) = \sigma'(C)$ (counting multiplicities); and*
- 2. $\emptyset \neq \sigma'(A_i) \cap \sigma'(C_j)$ for all i, j , where $A = \bigoplus A_i$ and $C = \bigoplus C_j$.*

Say G and H are *compatible* when such $A \in \mathcal{R}(G), C \in \mathcal{R}(H)$ exist.

Example

$\mathcal{R}(3K_1 \vee 2K_2)$ is projectionless.

Converse for complete/path components

Theorem (L-Oblak-Šmigoc 2020, 2021)

If $G = \bigcup_i G_i$ and $H = \bigcup_j H_j$ where each G_i, H_j is either a complete graph or a path, then $\mathcal{R}(G \vee H)$ contains a projection if and only if G and H are compatible.

Example

▶ $\mathcal{R}(2K_1 \vee 2K_2)$

▶ $\mathcal{R}(3K_1 \vee 2K_2)$

▶ $\mathcal{R}(4K_1 \vee 2K_2)$

Complete/path components as extreme cases

Theorem (L-Oblak-Šmigoc 2020, 2021)

Suppose $G = \bigcup_i G_i$ and $H = \bigcup_j H_j$ where G_i, H_j are connected.

1. Let $K_G = \bigcup_i K_{|G_i|}$ and $K_H = \bigcup_j K_{|H_j|}$.

If $\mathcal{R}(K_G \vee K_H)$ is projectionless, then so is $\mathcal{R}(G \vee H)$.

2. Let $P_G = \bigcup_i P_{|G_i|}$ and $P_H = \bigcup_j P_{|H_j|}$.

If $\mathcal{R}(P_G \vee P_H)$ contains a projection, then so does $\mathcal{R}(G \vee H)$.

Theorem (L-Oblak-Šmigoc 2020, 2021)

Suppose $G = \bigcup_i G_i$ and $H = \bigcup_j H_j$ where G_i, H_j are connected.

2. Let $P_G = \bigcup_i P_{|G_i|}$ and $P_H = \bigcup_j P_{|H_j|}$.

If $\mathcal{R}(P_G \vee P_H)$ contains a projection, then so does $\mathcal{R}(G \vee H)$.

Corollary (generalized Monfared-Shader)

If G and H are connected with $||G| - |H|| \leq 2$, then $\mathcal{R}(G \vee H)$ contains a projection, and this inequality is sharp.

Complete components: combinatorial characterisation

Theorem (LOS 2020)

Suppose $k \leq \ell$ and

$$G = \bigcup_{i=1}^k K_{n_i}, \quad H = \bigcup_{j=1}^{\ell} K_{m_j},$$

$$\iota(G) := |\{i : n_i = 1\}|, \quad \iota(H) := |\{j : m_j = 1\}|.$$

The following are equivalent:

1. $\mathcal{R}(G \vee H)$ is projectionless;
2. at least one of the following three conditions holds:
 - ▶ $|G| < \ell$;
 - ▶ $\iota(G) > 0$ and $k + \ell > \min\{|G| + \iota(G), |H| + \iota(H)\}$;
 - ▶ $\iota(G) = 0$ and $\iota(H) > 0$, and $k < \ell < 2k$ and $|H| < 2k$ and $|G| < k + \ell$.

Problem

Characterise compatibility for unions of paths.

Generic realisability

Questions

- ▶ Does it make any difference if we work over \mathbb{R} or \mathbb{C} ?
- ▶ Is compatibility sufficient for $\mathcal{R}(G \vee H)$ to contain a projection?
- ▶ There is a combinatorial characterisation of compatibility for two unions of complete graphs [LOS 2020]. What about unions of paths?
- ▶ If Γ is not a join (i.e., its complement is connected) and does not have the unique shortest path obstruction, can we determine whether $\mathcal{R}(\Gamma)$ contains a projection?

Thank you!

