Graph operator systems generated by a projection

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Graph operator systems

Let Γ be a connected graph with vertex set $\{1, 2, \ldots, n\}$.

eg

$\Gamma = P_3 = 1 - 2 - 3$

Graph operator system of Γ:

$\mathcal{S}(\Gamma) = \operatorname{span}\{e_{ij} \mid i = j \text{ or } ij \in E(\Gamma)\}$

$\quad = \{X \in M_n(\mathbb{C}) \mid x_{ij} = 0 \Rightarrow i \neq j \ \& \ ij \notin E(\Gamma)\}$

$\ast$

D_n - bimodule

$$\mathcal{S}(\Gamma) = \begin{bmatrix} \ast & \ast & 0 \\ \ast & \ast & \ast \\ 0 & \ast & \ast \end{bmatrix}$$

$\mathcal{R}(\Gamma) \subseteq \mathcal{S}(\Gamma)$

$\mathcal{R}(\Gamma) = \{X \in \mathcal{S}(\Gamma) \mid ij \in E(\Gamma) \Rightarrow x_{ij} \neq 0\}$

$$\mathcal{R}(\Gamma) = \begin{bmatrix} \ast & \cancel{\ast} & 0 \\ \cancel{\ast} & \ast & \cancel{\ast} \\ 0 & \cancel{\ast} & \ast \end{bmatrix}$$

$\cancel{\ast}$

$X \in \mathcal{R}(\Gamma) \iff \langle X, \underset{\ast}{D_n} \rangle = \mathcal{S}(\Gamma)$, as a D_n-bimodule.

Question

For which graphs Γ does $\mathcal{R}(\Gamma)$ contain a projection?

IEPG

Some simple reformulations

Proposition

TFAE:

1. $\mathcal{R}(\Gamma)$ *contains a projection;*
2. $\mathcal{R}(\Gamma)$ *contains a Hermitian unitary;*
3. $\mathcal{R}(\Gamma)$ *contains a normal matrix with at most two distinct eigenvalues.*

$$X \in \mathcal{R}(\Gamma) \implies \underline{\alpha X + \beta I} \in \mathcal{R}(\Gamma)$$

$$\alpha \neq 0 \quad \beta \in \mathbb{C}$$
$$\alpha \in \mathbb{C}$$

$$P \longmapsto 2P - I$$

Paths

$$P_n = (1) - (2) - \cdots - (n) \qquad \mathcal{R}(P_n) = \begin{bmatrix} * & * & & 0 \\ * & * & * & \\ & * & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

Proposition (folklore)

If $X \in \mathcal{R}(P_n)$ is normal, then X has n distinct eigenvalues.
In particular, if $n \geq 3$, then $\mathcal{R}(P_n)$ is projectionless.

Pf $X \in \mathcal{R}(P_n)$

$$\Rightarrow I, \underset{\begin{pmatrix} * & * & 0 & \cdots \\ * & \ddots & & \\ 0 & & \ddots & \\ \vdots & & & \ddots \end{pmatrix}}{X}, \underset{\begin{pmatrix} * & * & * & \\ * & \ddots & & \ddots \\ * & \ddots & \ddots & \\ & \ddots & & \end{pmatrix}}{X^2}, \ldots, X^{n-1} \text{ are lin indep}$$

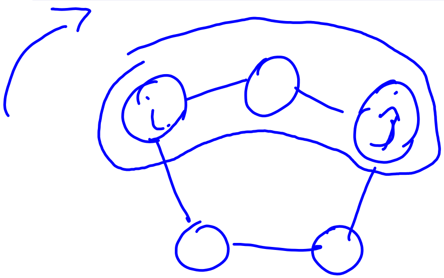
$\Rightarrow$ min poly of X has degree n,

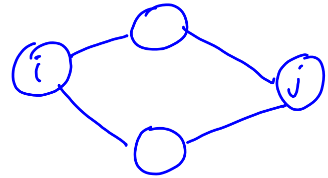
$\Rightarrow$ X has n distinct eigenvalues. $\blacksquare$

Unique shortest path obstruction

Proposition (Fallat et al. 2013)

If $\mathrm{dist}_\Gamma(i,j) \geq 2$ *and this is attained by a unique path in* Γ*, then* $\mathcal{R}(\Gamma)$ *is projectionless.*





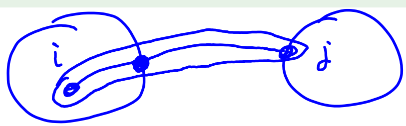
$\mathcal{R}(C_5) \not\ni$ projection

$\mathcal{R}(C_4) \ni$ projection

Example

If Γ has a cut edge and at least 3 vertices, then $\mathcal{R}(\Gamma)$ is projectionless.

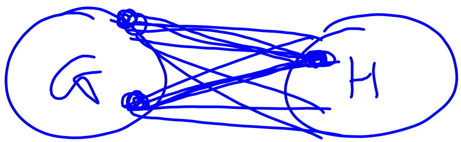
Γ :



The join of two graphs

Let G, H be any graphs (possibly disconnected).

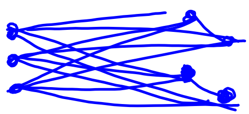
The join $G \vee H$ is connected and usually lacks the unique shortest path obstruction.



no 0's

$$\mathcal{R}(G \vee H) = \left[\begin{array}{c|c} \mathcal{R}(G) & * \\ \hline * & \mathcal{R}(H) \end{array}\right]$$

eg/ $3K_1 \vee 2K_2$



$$\mathcal{R}(3K_1 \vee 2K_2) = \left[\begin{array}{c|c} \begin{smallmatrix} * & & \\ & * & \\ & & * \end{smallmatrix} & * \\ \hline * & \begin{smallmatrix} * & * & 0 \\ * & * & \\ 0 & * & * \end{smallmatrix} \end{array}\right]$$

Theorem (Monfared-Shader 2016)

If G and H are connected graphs with $|G| = |H|$, then $\mathcal{R}(G \vee H)$ contains a projection.

Key: $\mathcal{R}(G)$ & $\mathcal{R}(H)$ admit matrices with distinct eigenvalues, & no eigenvectors with a 0 entry

Necessary condition (connected G, H)

Write $\sigma'(X) = \sigma(X) \setminus \{0, 1\}$.

Proposition (L-Oblak-Šmigoc 2020)

If G and H are connected and $\mathcal{R}(G \vee H)$ contains a projection, then there exist positive semidefinite contractions $A \in \mathcal{R}(G)$ and $C \in \mathcal{R}(H)$ so that $\emptyset \neq \sigma'(A) = \sigma'(C)$.

counting multiplicities

Pf Let $P = \begin{pmatrix} A & B \\ B^* & I-C \end{pmatrix} \in \underline{\mathcal{R}(G \vee H)}$

be a projection

① $A \in \mathcal{R}(G)$, $C \in \mathcal{R}(H)$ are psd contractions

& $B \neq 0 \Rightarrow \underline{\sigma'(A), \sigma'(C)} \neq \emptyset$.

② Compress blocks to eigenspaces for $\sigma'(A), \sigma'(C)$

$P \supseteq P_0 = \begin{pmatrix} A_0 & B_0 \\ B_0^* & I-C_0 \end{pmatrix}$, a projection, & $\sigma(A_0) = \sigma'(A)$, $\sigma(C_0) = \sigma'(C)$

③ $P_0 = \begin{pmatrix} A_0 & B_0 \\ B_0^* & I - C_0 \end{pmatrix}$;

$P_0^2 = P_0 \Rightarrow$ (a) B_0 invertible

& $\sigma(A_0), \sigma(C_0) \subseteq (0,1)$

& (b) $A_0 B_0 = B_0 C_0$

$\Rightarrow \sigma(A_0) = \sigma(C_0)$ $\boxed{3}$

Necessary condition (arbitrary G, H)

Theorem (L-Oblak-Šmigoc 2020)

If $G = \bigcup G_i$ and $H = \bigcup H_j$ where G_i, H_j are connected, and $\mathcal{R}(G \vee H)$ contains a projection, then there exist positive semidefinite contractions $A \in \mathcal{R}(G)$ and $C \in \mathcal{R}(H)$ so that

1. $\sigma'(A) = \sigma'(C)$ *(counting multiplicities); and*
2. $\emptyset \neq \sigma'(A_i) \cap \sigma'(C_j)$ *for all i, j, where $\boxed{A = \bigoplus A_i}$ and $C = \bigoplus C_j$.*

Say G and H are *compatible* when such $A \in \mathcal{R}(G)$, $C \in \mathcal{R}(H)$ exist.

$$M_A = \begin{array}{c|c} & A_1 \; \dots \; A_k \\ \hline \begin{matrix} \lambda_1 \\ \vdots \\ \lambda_r \end{matrix} & \begin{bmatrix} & & \\ & m_{ij}(A) & \\ & & \end{bmatrix} \end{array} \; ; \quad M_C = \begin{array}{c|c} & C_1 \; \dots \; C_\ell \\ \hline \begin{matrix} \lambda_1 \\ \vdots \\ \lambda_r \end{matrix} & \begin{bmatrix} & & \\ & m_{ij}(C) & \\ & & \end{bmatrix} \end{array}$$

$$m_{ij}(A) = \mathrm{mult}(\lambda_i, A_j)$$

Compatibility: ① row sums of M_A, M_C are equal

② no col of $M_A \perp$ a col of M_C

Example

$\mathcal{R}(3K_1 \vee 2K_2)$ is projectionless.

Pf $3K_1$

$$\begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$$

$$\mathcal{R}(2K_2) = \left(\begin{array}{cc|cc} * & * & \multicolumn{2}{c}{0} \\ * & * & & \\ \hline \multicolumn{2}{c|}{0} & * & * \\ & & * & * \end{array}\right)$$

Compatible M_C

$$\begin{array}{cc} K_2 & K_2 \\ \multicolumn{2}{c}{\begin{bmatrix} \quad & \quad \end{bmatrix}} \end{array}$$

M_A could be

$$\cancel{\begin{array}{cccc} & K_1 & K_1 & K_1 \\ \lambda & [1 & 1 & 1] \end{array}} \quad \text{row sum } 3$$

$$\begin{array}{c} \lambda \\ \mu \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{not compat}$$

$\perp$ cols necc.

Converse for complete/path components

Theorem (L-Oblak-Šmigoc 2020, 2021)

If $G = \bigcup_i G_i$ and $H = \bigcup_j H_j$ where each G_i, H_j is either a complete graph or a path, then $\mathcal{R}(G \vee H)$ contains a projection if and only if G and H are compatible.

Idea: take compatible $A \in \mathcal{R}(G)$,
$C \in \mathcal{R}(H)$;

try to complete $\left(\begin{array}{c|c} A & B \\ \hline B^* & I-C \end{array}\right)$ to a projection, where $b_{ij} \neq 0$.

Diagonalize:

$$Q \oplus \bigoplus_{\lambda_i \in \sigma'(A) = \sigma'(C)} \begin{pmatrix} \lambda_i I_{n_i} & * \\ * & (1-\lambda_i) I_{n_i} \end{pmatrix}$$

Q: diag projection

Sufficient evecs of A, C can be chosen "generically"

Easy for K_n

Harder for P_n

Example

- $\mathcal{R}(2K_1 \vee 2K_2)$ $\ni$ projection

$$M_A = \lambda \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad M_C = \lambda \begin{array}{c} \begin{matrix} K_2 & K_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}$$

- $\mathcal{R}(3K_1 \vee 2K_2)$ $\not\ni$ projection

- $\mathcal{R}(\underline{4K_1} \vee 2K_2)$ $\ni$ projection

$$M_A = \begin{matrix} \lambda \\ \mu \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} ; \qquad M_C = \begin{matrix} \lambda \\ \mu \end{matrix} \begin{array}{c} \begin{matrix} K_2 & K_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

Complete/path components as extreme cases

Theorem (L-Oblak-Šmigoc 2020, 2021)

Suppose $G = \bigcup_i G_i$ *and* $H = \bigcup_j H_j$ *where* G_i, H_j *are connected.*

1. *Let* $\underline{K_G = \bigcup_i K_{|G_i|}}$ *and* $\underline{K_H = \bigcup_j K_{|H_j|}}$.
   If $\underline{\mathcal{R}(K_G \vee K_H)}$ *is projectionless, then so is* $\underline{\mathcal{R}(G \vee H)}$.
2. *Let* $\underline{P_G = \bigcup_i P_{|G_i|}}$ *and* $\underline{P_H = \bigcup_j P_{|H_j|}}$.
   If $\underline{\mathcal{R}(P_G \vee P_H)}$ *contains a projection, then so does* $\underline{\mathcal{R}(G \vee H)}$.

Theorem (L-Oblak-Šmigoc 2020, 2021)

Suppose $G = \bigcup_i G_i$ and $H = \bigcup_j H_j$ where G_i, H_j are connected.

2. *Let $P_G = \bigcup_i P_{|G_i|}$ and $P_H = \bigcup_j P_{|H_j|}$.*
   If $\mathcal{R}(P_G \vee P_H)$ contains a projection, then so does $\mathcal{R}(G \vee H)$.

Corollary (generalized Monfared-Shader)

If G and H are connected with $\big||G| - |H|\big| \leq 2$, then $\mathcal{R}(G \vee H)$ contains a projection, and this inequality is sharp.

$$M_A = \begin{bmatrix} \vdots \end{bmatrix} \ell \overset{?}{=} M_C = \begin{bmatrix} \vdots \end{bmatrix} m$$

$$\ell = m \iff \big||G| - |H|\big| \leq 2$$

Complete components: combinatorial characterisation

Theorem (LOS 2020)

Suppose $k \leq \ell$ and

$$G = \bigcup_{i=1}^{k} K_{n_i}, \quad H = \bigcup_{j=1}^{\ell} K_{m_j},$$

$$\iota(G) := |\{i : n_i = 1\}|, \quad \iota(H) := |\{j : m_j = 1\}|.$$

The following are equivalent:

1. $\mathcal{R}(G \vee H)$ *is projectionless;*
2. *at least one of the following three conditions holds:*
   - $|G| < \ell$;
   - $\iota(G) > 0$ *and* $k + \ell > \min\{|G| + \iota(G), |H| + \iota(H)\}$;
   - $\iota(G) = 0$ *and* $\iota(H) > 0$, *and* $k < \ell < 2k$ *and* $|H| < 2k$ *and* $|G| < k + \ell$.

Problem

Characterise compatibility for unions of paths.

Generic realisability

Questions

- Does it make any difference if we work over $\mathbb{R}$ or $\mathbb{C}$?
- Is compatibility sufficient for $\mathcal{R}(G \vee H)$ to contain a projection?
- There is a combinatorial characterisation of compatibility for two unions of complete graphs [LOS 2020]. What about unions of paths?
- If Γ is not a join (i.e., its complement is connected) and does not have the unique shortest path obstruction, can we determine whether $\mathcal{R}(\Gamma)$ contains a projection?

Thank you!