EXTENSION THEORY AND VNA

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RECALL: B C^*-ALG., $M(B)$ MULTIPLIER

$M(B)/B = C(B)$ CORONA ALG.

REM: 1) IF $B = K(H)$, THEN $M(K) = B(H)$

$$M(K)/K = B(H)/K = Q(H)$$

CALKIN ALG.

2) THM: A C^*-ALG.

$\exists$ UNITAL C^*-ALG $M(A)$ ST.

$A \triangleleft M(A)$ ESSENTIAL IDEAL

(i.e. $A \cap J \neq \{0\}$, $\forall J \triangleleft M(A)$, $J \neq \{0\}$)

AND UNIVERSAL;

$$\begin{tikzcd} D \arrow[r, dashed, "\exists ! \varphi"] & M(A) \\ A \arrow[u, phantom, "\triangledown"] \arrow[r, "id"] & A \arrow[u, phantom, "\triangledown"] \end{tikzcd}$$

$\ker \varphi = A^{\perp} = \{x \in D ;\ Ax = 0\}$

THM: (VOICULESCU, 1976)

H sep. HILBERT

A UNITAL, SEP C^*-ALG OF $Q(H)$.

THEN

$$A = \left(A' \cap Q(H)\right)' \cap Q(H).$$

RECALL: 1) TO ANY EXT. $0 \to B \to E \to A \to 0$

IS ASSOCIATED ITS BUSBY INVARIANT

$$\sigma : A \to M(B)/B \quad *\text{-HOMOM.}$$

2) AN EXT. IS DETERMINED BY ITS BUSBY INV. (UP TO STRONG ISOM.)

3) AN EXT IS ESSENTIAL (ie. $B \triangleleft E$ ESSEN) IFF BUSBY INV. INJECTIVE

4) AN EXT. $\sigma : A \to M(B)/B$ IS TRIVIAL IF

$$\begin{tikzcd} & M(B) \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\exists \sigma_0"] \arrow[r, "\sigma"] & M(B)/B \end{tikzcd}$$

σ_0 $*$-HOMOM

$\sigma = \pi \circ \sigma_0$.

REM σ TRIVIAL IFF $0 \to B \to E \underset{q}{\overset{s}{\rightleftarrows}} A \to 0$

s $*$-HOMOM ST $q \circ s = 1_A$.

5) AN EXT. $\tau : A \to M(B)/B$ ABSORBING.

IF τ STRONG. EQUIV. TO $\tau \oplus \sigma$

FOR ANY TRIVIAL EXT. σ

THM (VOICULESCU, 1976) A SEP. C^*-ALG.

EVERY ESSENTIAL (NON-UNITAL) EXT OF A BY K IS ABSORBING.

IF A UNITAL, SEP. C^*-ALG., EVERY ESSENTIAL UNITAL EXT OF A IS UNITAL ABSORBING.

Q? HOW TO GENERALIZE THE RELATIVE BICOMMUTANT QUESTION?

TWO DIRECTIONS:

* 1) TOWARDS GERT PEDERSEN'S QUEST.
* 2) TYPE II_∞ vN FACTOR CASE.

FOR (1): Q? (PEDERSEN, 1988)

B σ-UNITAL C^*-ALG ST. $M(B)/B$ SIMPLE

A UNITAL, SEP. SUB-C^*-ALG OF $M(B)/B$

$$(A' \cap C(B))' \cap C(B) = A \quad ?$$

NEED (MORE) GENERAL VERSION OF THE ABSORPTION THM.

ELLIOTT - KUCEROVSKY (2001)

ABSTRACT VOICULESCU - BDF ASORPTION THM.

DEF: A, B SEP. C^*-ALG, B STABLE

AN ESSENTIAL EXT. $\varphi : A \to C(B)$ IS <u>TRIVIAL IN THE NUCLEAR SENSE</u> IF

$$\begin{tikzcd} & M(B) \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\varphi_0"] \arrow[r, "\varphi"] & C(B) \end{tikzcd}$$

$\varphi_0 : A \to M(B)$ WEAKLY NUCLEAR.

(ie. $\forall\, b \in B$, CONTRACTION, $a \in A \mapsto b\varphi_0(a)b^* \in B$ IS A NUCLEAR MAP.)

DEF: $B \triangleleft C$ PROPER IDEAL OF C.

THEN C IS <u>PURELY LARGE WRT B</u>.

IF $\forall\, c \in C \setminus B$, $\mathrm{her}(c) = \overline{cBc^*}$ CONTAINS A STABLE, FULL SUB-ALG OF B.

(ie. $D \subset B$ INCLUSION OF C^*-ALG.

D FULL IN B IF $\overline{\mathrm{Span}\{BDB\}} = B$).

THM: (ELLIOTT- KUCEROVSKY, 2001.
FOR THE UNITAL CASE.
GABE, 2016 FOR THE NON-UNITAL CASE)

A, B SEP. C^*-ALG., B STABLE

$\varphi : A \to C(B)$ ESSENTIAL EXT. ST.

$\varphi \oplus 0 \sim \varphi$ (RESP. φ UNITAL).

THEN φ ABORBS EVERY WEAKLY NUCL. (resp. STRONGLY UNITAL) TRIVIAL EXT. OF B BY A.

$$\Longleftrightarrow$$

$\pi^{-1}(\varphi(A))$ PURELY LARGE WRT B

THM (T.G - PING NG).

$B \cong K$ OR B SEP. SIMPLE, STABLE, P.I

A SEP. UNITAL SUB-C^*-ALG OF $C(B)$.

THEN $$(A' \cap C(B))' \cap C(B) = A.$$

REM (LIN, 1991) B σ-UNITAL, STABLE, SIMPLE

$M(B)/B$ SIMPLE IFF $B \cong K$ OR B P.I.

THM. (T.G.-P.Ng)

B σ-UNITAL, STABLE, SIMPLE C^*-ALG.

A SEP. UNITAL SUB-C^*-ALG OF $C(B)$

THE $\left(A' \cap C(B)\right)' \cap C(B) = A$.

(STABLE CASE OF PEDERSEN QUESTION).

THM. (T.G - V. KAFTAL - P. NG). A, B C^*-ALG.

A SEP., B σ-UNITAL, STABLE

$\varphi : A \to C(B)$ ESSENTIAL EXT. ST.

$\varphi \oplus 0 \sim \varphi$ (resp. φ UNITAL)

THEN φ ABSORBS EVERY WEAKLY NUCL. (resp. STRONGLY UNITAL) TRIVIAL EXT. OF A BY A

$\Longleftrightarrow$

$\pi^{-1}(\varphi(A))$ PURELY LARGE WRT B.

SECOND DIRECTION:

A VOICULESCU - BDF ABSORPTION THM. FOR TYPE II_{∞} FACTORS.

M vNa.

DEF: BREUER IDEAL K_M OF M. IS THE NORMED CLOSED TWO-SIDED IDEAL OF M GENERATED BY THE FINITE PROJECTIONS OF M.

FACTS:

① M σ-FINITE, PROPERLY INF. FACTOR (ie. M TYPE I_{∞} OR II_{∞})

THE K_M UNIQUE PROPER IDEAL OF M.

IF $M = B(H)$, THEN $K_M = K$.

② M σ-FINITE, THEN $M(K_M) = M$.

③ M TYPE II_{∞} FACTOR

a) K_M NOT σ-UNITAL

b) K_M NOT STABLE.

RECALL: (HJELMBORG - RØRDAM)

EQUIVALENT CHARACTERIZATION OF STABILITY FOR σ-UNITAL C*-ALGB.

(HR) $\forall a \in B_+$, $\forall \varepsilon > 0$, $\exists x \in B$ ST.

$\| a - x^*x \| < \varepsilon$ AND $\| a x^* x \| < \varepsilon$.

FACTS: ① B STABLE $\Rightarrow$ B HAS (HR)

② B σ-UNITAL, HAS (HR) $\Longrightarrow$ B STABLE

③ IF M TYPE II_∞ FACTOR, THEN $\underline{K_M \text{ HAS (HR)}}$.

④ IF M TYPE II_∞ FACTOR, THEN

$\forall e \in (K_M)_+$, RANGE PROJ. R_e INFINITE,

$\mathrm{her}(e) = \overline{e K_M e}$ σ-UNITAL, HAS (HR).

HENCE STABLE

MOREOVER,

$$M(\mathrm{her}(e)) = \{ b \in R_e M R_e \mid be \in \mathrm{Her}(e) \text{ AND } eb \in \mathrm{her}(e) \}$$

(5) B C^*-ALG. WITH COND (HR)

$S \subset B$ COUNTABLE SUBSET

THEN $\exists$ SEP. STABLE C^*-SUBALG

$D \subseteq B$ WITH $S \subseteq D$.

NEED TO WEAKEN THE NOTION OF WEAK NUCL.

FOR A CP MAP $\Phi: A \longrightarrow M = M(K_M)$.

DEF: B UNITAL C^*-ALG. $B \subseteq B(H)$.

$A \triangleleft B$; $S \subset B$ COUNTABLE

A SEQ OF POS. CONTRACTIONS $(e_n)_{n \geq 1}$ IN A

IS A S-QUASICENTRAL SOT-APPROX UNIT

IF 1) $e_{n+1} e_n = e_n$, $\forall n \geq 1$

2) $e_n \nearrow 1$ IN SOT

3) $\| [e_n, s] \| \longrightarrow 0$, $\forall s \in S$.

<u>DEF.</u> : M σ-FINITE II_∞ FACTOR

A SEP. C^*-ALG.

1) A CP MAP $\Phi : A \to M$ IS <u>SOT-WEAKLY NUCLEAR</u> IF

$\exists$ SOT-APPROX UNIT $(e_n)_{n \geq 1}$ FOR K_M. ST.

* $(e_n)_{n \geq 1}$ QUASICENTRALIZES $\Phi(A)$

* $\forall n \geq 1$, $e_n \Phi(\cdot) e_n : A \to M$ NUCLEAR

2) A <u>TRIVIAL</u> EXT. $\varphi : A \longrightarrow M/K_M$ IS <u>SOT-WEAKLY NUCLEAR</u> IF

$$\exists\ \Phi \qquad \begin{array}{ccc} & & M \\ & \nearrow & \downarrow \pi \\ A & \xrightarrow{\varphi} & M/K_M \end{array}$$

Φ SOT-WEAKLY NUCL.

$*$-HOMOM.

<u>THM</u> (T.G. - VICTOR K. - PING. NG).

M σ-FINITE TYPE II_∞ FACTOR

A SEP C^*-ALG.

$\varphi : A \longrightarrow M/\mathcal{K}_M$ ESSENTIAL EXT

s.t. $\varphi \oplus 0 \sim \varphi$ (Resp. φ UNITAL)

THEN φ ABSORBS EVERY (Resp. STRONGLY UNITAL) SOT - WEAKLY NUCLEAR TRIVIAL EXTENSION.

<u>THM</u>: M σ-FINITE II_∞ FACTOR

A SEP. UNITAL C^*-SUBALG OF $M/\mathcal{K}_M$

THEN $(A' \cap \mathcal{K}_M)' \cap \mathcal{K}_M = A$.
