The Norm of L-matrices

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The Ambient Space

The first Hilbert space

$$\ell^2 := \{x : \|x\|_2 < \infty\},$$

where

$$x = (x_1, x_2, \dots)$$

and

$$\|x\|_2 = (|x_1|^2 + |x_2|^2 + \cdots)^{1/2}.$$

The Ambient Space

The Hilbert–Hardy space

$$H^2 := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_2 < \infty\},$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}.$$

An infinite matrix

To each bounded operator $A : \ell^2 \to \ell^2$ corresponds a unique infinite matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$a_{ij} = \langle A\mathrm{e}_j, \mathrm{e}_i \rangle.$$

Some questions

- Given an infinite matrix A, does it represent a bounded operator on ℓ^2?
- Find $\|A\| := \sup \|Ax\|_2 / \|x\|_2$.
- If the precise value of $\|A\|$ is not known, find an estimate.

The Hilbert matrix

Let

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Hilbert matrix

Hilbert (1894) introduced this matrix to study a question in approximation theory. He obtained the determinant of H_n and showed that

$$\|H\|_{\ell^2\to\ell^2} = \pi.$$

We also know that

$$\sigma_p(H) = \emptyset \quad \text{and} \quad \sigma(H) = [0, \pi].$$

D. Hilbert, Ein Beitrag zur Theorie des Legendreschen Polynoms, Acta Mathematica, 18:155–159, 1894.

M. Choi, Tricks or treats with the Hilbert matrix, Amer. Math. Monthly, 90(5):301–312, 1983.

The Hankel matrix

$$H = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Hankel operator

For $\varphi \in L^\infty(\mathbb{T})$, the *Hankel operator*

$$H_\varphi : H^2 \longrightarrow \overline{H_0^2}$$

is defined by

$$H_\varphi f := (I - P)(\varphi f),$$

where P is the Riesz projection of L^2 onto H^2.

Matrix representation of H_φ

$$H_\varphi = \begin{pmatrix} \hat{\varphi}(-1) & \hat{\varphi}(-2) & \hat{\varphi}(-3) & \cdots \\ \hat{\varphi}(-2) & \hat{\varphi}(-3) & \hat{\varphi}(-4) & \cdots \\ \hat{\varphi}(-3) & \hat{\varphi}(-4) & \hat{\varphi}(-5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\hat{\varphi}(n) := \int_{\mathbb{T}} \varphi(\zeta)\bar{\zeta}^n \, dm(\zeta), \qquad n \in \mathbb{Z}.$$

Nehari Theorem

Theorem

Let $\varphi \in L^2(\mathbb{T})$. *Then*

$$\|H_\varphi\| = dist(\varphi, H^\infty) = \|\varphi - H^\infty\|.$$

Moreover, there is $\psi \in H^\infty$ *such that*

$$\|H_\varphi\| = \|\varphi - \psi\|_\infty.$$

Note that $H_\varphi = H_{\varphi-\psi}$.

Z. Nehari, On bounded bilinear forms, Ann. of Math., 65(2):153–62, 1957.

The Cesàro matrix

Let

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Cesàro operator I

Clearly

$$C\left((a_1, a_2, a_3, \dots)\right) = \left(a_1, \frac{a_1 + a_2}{2}, \frac{a_1 + a_2 + a_3}{3}, \dots\right).$$

The Cesàro operator II

Another interpretation: C acts on H^2 by

$$(Cf)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{j=0}^{n} a_j \right) z^n.$$

An equivalent formulation is

$$(Cf)(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w}\, dw.$$

Hardy's inequality

For $a_n \geq 0$,

$$\sum_{n=1}^{N} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2 \leq 16 \sum_{n=1}^{N} a_n^2.$$

The Cesàro matrix

We know that

$$\|C\|_{\ell^2 \to \ell^2} = 2$$

and that

$$\sigma_p(C) = \emptyset \quad \text{and} \quad \sigma(C) = \{z : |z-1| \le 1\}.$$

A. Brown, P. Halmos, A. Shields, Cesàro operators, Acta Sci. Math. (Szeged), 26: 125–137, 1965.

An L-matrix

Let

$$L = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An L-matrix

We show that

$$\|L\|_{\ell^2 \to \ell^2} = 4.$$

The L-matrix

Let

$$L_s = \begin{pmatrix} \frac{1}{s} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The L-matrix

Here, we show that

$$\|L_s\|_{\ell^2\to\ell^2} = 4, \qquad s \geq \frac{1}{2} \text{ (at least).}$$

Multipliers

$X =$ a Banach space of analytic functions on the open unit disc $\mathbb{D}$.

Characterizing $\mathrm{Mult}(X)$, the multipliers of X, is essential in various studies of function spaces, e.g., zero sets, invariant subspaces, cyclic elements, etc.

Multipliers

JM–Ransford (2019) observed that $h(z) = \sum_{n=0}^{\infty} c_n z^n$ is a Hadamard multiplier for the superharmonically weighted Dirichlet Space $\mathcal{D}_w$ if and only if the infinite matrix

$$\begin{pmatrix} c_1 & c_2 - c_1 & c_3 - c_2 & \cdots \\ 0 & c_2 & c_3 - c_2 & \cdots \\ 0 & 0 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

acts as a bounded operator on ℓ^2.

Multipliers

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acts as a bounded operator on ℓ^2.

C-matrices

Fix a sequence $(a_n)_{n\geq 0}$. This observation naturally leads to studying C-matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

C-matrices

or

$$C = C[a_n] = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & \cdots \\ a_2 & a_2 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

L-matrices

and L-matrices

$$L = L[a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_1 & a_2 & \cdots \\ a_2 & a_2 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A necessary condition

Since

$$\|Ae_n\|_2^2 = (n+1)|a_n|^2 + |a_{n+1}|^2 + |a_{n+1}|^2 + \cdots ,$$

a necessary condition is

$$a_n = O\left(\frac{1}{\sqrt{n}}\right), \qquad \text{as } n \to \infty.$$

Sharpness, lacunary sequences

If

$$a_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad n = 4^k,\ k \geq 0, \\ 0 & \text{if} \quad \text{otherwise}, \end{cases}$$

then

$$\|A\| < \infty.$$

Not sufficient

Fix any $0 < \alpha < 1$. If

$$a_n = \frac{1}{(n+1)^\alpha}, \qquad n \geq 0,$$

then

$$\|A\| = \infty.$$

A sufficient condition

If

$$a_n = O\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty,$$

then

$$\|A\| < \infty.$$

Decreasing sequences

Theorem (JM-Bouthat 2020)

Let $A = [a_n]$ be an L-matrix such that

$$a_0 > a_1 > a_2 > \cdots > 0,$$

and that

$$\Delta := \sup_{n \geq 1} \frac{2a_n(a_n + a_{n-1})}{a_{n-1} - a_n} < \infty.$$

Then A is a bounded operator on ℓ^2 and, moreover,

$$\|A\| \leq \max\{2a_0, \Delta\}.$$

The L_s matrix

Corollary

Let

$$L_s = \begin{pmatrix} \frac{1}{s} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\|L_s\| = 4, \qquad s \geq \frac{1}{2}.$$

An open question

Find

$$s_0 = \inf\{s : \|L_s\| = 4\}.$$

Clearly,

$$\frac{1}{4} < s_0 \leq \frac{1}{2}.$$

A recent work

According to a recent (unpublished) work,

$$\frac{\sqrt{6(8+3\sqrt{3})-\sqrt{3}-3}}{12} \leq s_0 \leq \frac{1}{2\sqrt{2}}.$$

In other words,

$$0.347 \leq s_0 \leq 0.356.$$

General case

Theorem (JM-Bouthat 2020)

Let $A = [a_n]$ be an L-matrix. Suppose that there is a strictly decreasing sequence $(\delta_n)_{n\geq 0}$ such that

$$\delta_0 > \delta_1 > \delta_2 > \cdots > 0$$

and that

$$\Delta := \sup_{n\geq 1} \frac{(|a_n| + \delta_n)(|a_n| + \delta_{n-1})}{\delta_{n-1} - \delta_n} < \infty.$$

Then A is a bounded operator on ℓ^2 and, moreover,

$$\|A\| \leq \max\{\delta_0 + |a_0|, \Delta\}.$$

The sufficient condition

Corollary

If

$$a_n = O\left(\frac{1}{n}\right), \qquad \text{as } n \to \infty,$$

then

$$\|A\| < \infty.$$

The sufficient condition

- There are further results about the 2-norm of C-matrices.
- Detailed study of C-matrices and L-matrices created by lacunary sequences.
- Similar results for p-norms of C-matrices and L-matrices.
- F. Štampach (2021) has studied the spectral analysis of L_s matrices.

Thank You For Your Attention