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Semi doubly stochastic operators on $L^1(X)$ and its quantum application

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Presentation Outline

Short History

- **1903: Muirhead**

Muirhead's inequality

Let the components of $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be non-negative integer.

1. $$\sum_{\pi} \alpha_{\pi(1)}^{x_1} \alpha_{\pi(2)}^{x_2} \cdots \alpha_{\pi(n)}^{x_n} \leq \sum_{\pi} \alpha_{\pi(1)}^{y_1} \alpha_{\pi(2)}^{y_2} \cdots \alpha_{\pi(n)}^{y_n};$$

2. the sum of the k largest components of X is less than or equal to the sum of the k largest components of Y, $k = 1, 2, \ldots, n-1$ with equality when $k = n$.

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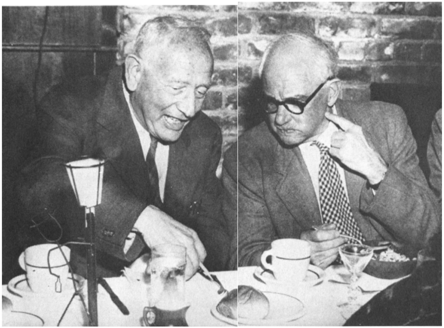

Definition of Vector Majorization

Let $X = (x_1, x_2, \ldots, x_n)$ be a real vector. X has been reordered so that $x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow$.

Definition 1 (1929- Hardy, Littlewood and Pólya [3])

if $X, Y \in \mathbb{R}^n$, we say X **is *majorized* by** Y, dtenoted $X \prec Y$, if

$$\begin{aligned} x_1^\downarrow &\leq y_1^\downarrow \\ x_1^\downarrow + x_2^\downarrow &\leq y_1^\downarrow + y_2^\downarrow \\ &\vdots \\ \sum_{j=1}^{k} x_j^\downarrow &\leq \sum_{j=1}^{k} y_j^\downarrow, \quad k \in \{1, \ldots, n-1\} \\ \textit{and} \quad \sum_{j=1}^{n} x_j &= \sum_{j=1}^{n} y_j. \end{aligned}$$

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Equivalent Conditions for Vector Majorization

Theorem 2 (1934- Hardy, Littlewood, and Pólya [3])

For $X, Y \in \mathbb{R}^n$ the followings are equivalent.

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Why l^1 space?

$$l^1 = \{f : \mathbb{N} \to \mathbb{R} : \quad \sum_{n \in \mathbb{N}} |f(n)| < +\infty\}.$$

Presentation Outline

In 1999, Nielsen used vector majorization to link problem of state transformation with mathematics in a finite dimensional system.

Any isolated physical system is identified with some finite or infinite dimensional Hilbert spaces and its pure states, which system can be described completely by one of them, correspond to unit vectors (for details see [6, section 2.2.1]).

The state space of a composite system is modelled by the tensor product of subsystems (see [6, section 2.2.8]).

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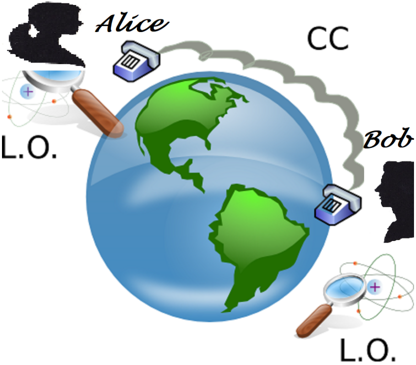
Local Operations and Classical Communication

The parties are not allowed to exchange particles coherently.
Only local operations and classical communication is allowed.



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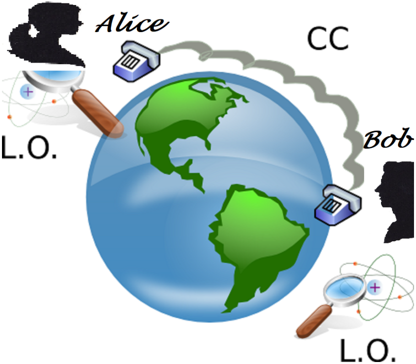

Nielsen's Theorem in the finite dimensional

Theorem 3 (Nielsen's Theorem [6])

$|\psi\rangle$ can be converted to $|\phi\rangle$ by LOCC channel if and only if $\lambda_\psi \prec \lambda_\phi$.

Theorem 4 (Schmidt decomposition; infinite case)

For every $|\psi\rangle \in H_a \otimes H_b$ there exist orthonormal Schmidt sets (not necessarily basis) $\{|e_i\rangle\}_{i=1}^{\infty} \subset H_a$ and $\{|f_i\rangle\}_{i=1}^{\infty} \subset H_b$ s.t

$$|\psi\rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i}|e_i\rangle \otimes |f_i\rangle,$$

where

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$$l^1 = \{f : \mathbb{N} \to \mathbb{R} : \quad \sum_{n\in\mathbb{N}} |f(n)| < +\infty\}$$

Since the space of all real-valued integrable functions $L^1(X, \mu)$ are used in the theoretical discussion of problems in various field of science such as finance, engineering, physics, statistics, and other disciplines,we prefer to work more generally on L^1 space. It is clear that for σ-finite measure space $(\mathbb{N}, \mu)$, when μ is the counting measure, L^1 and l^1 coincide.

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$$Tf \prec f \quad \forall f \in L^1(X).$$

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Decreasing Rearrangement for Integrable Functions

Definition 5 (Chong-[2])

If f is any measurable function

$$f^{\downarrow}(s) = \inf\{t : d_f(t) \leq s\}, \qquad 0 \leq s \leq \mu(X) \tag{1}$$

where $d_f(t) = \mu\{x : f(x) > t\}$ for all $t \in \mathbb{R}$.

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Definition of Majorization Based on Decreasing Rearrangement

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$$\int_0^s f^{\downarrow} dm \le \int_0^s g^{\downarrow} dm \quad \forall\, 0 \le s \le \mu(X)$$

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Doubly stochastic operators

Theorem 7 (Ryff-[7])

For a finite measure space (X, A, μ), a bounded linear operator T on $L^1(X,\mu)$ satisfies $Tf \prec f$, for each $f \in L^1(X,\mu)$, if and only if

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for all $f \in L^1(X,\mu)$ and $g \in L^\infty(X,\mu)$, where $T^* : L^\infty(X,\mu) \to L^\infty(X,\mu)$ is the adjoint operator of T.

A bounded linear operator on $L^1(X,\mu)$ which satisfies the equivalent conditions of the above theorem is called a doubly stochastic operator, that we will denote the class of all doubly stochastic operators on $L^1(X,\mu)$ by $\mathcal{DS}(L^1(X,\mu))$.

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Example 8

Let $D : l^1 \to l^1$ be right shift operator, it is easily seen that

$$\forall f \in l^1 \quad Df \prec f.$$

Now let

$$f := \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \quad then \quad Df = \sum_{n=1}^{\infty} \frac{1}{2^n} e_{n+1}$$

Hence

$$0 = (Df)_1 = \sum_{n=1}^{\infty} \langle De_n, e_1 \rangle f_n = \sum_{n=1}^{\infty} \frac{\langle De_n, e_1 \rangle}{2^n}$$

$$\Rightarrow \forall n \in \mathbb{N}, \quad \langle De_n, e_1 \rangle = 0$$

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$$\Rightarrow \forall n \in \mathbb{N}, \quad \langle De_n, e_1 \rangle = 0$$

So D is not a doubly stochastic.

Semi-doubly stochastic on $L^1(X)$

Definition 9

Let (X, μ) be σ finite measure space. A positive operator T on $L^1(X)$ is called semi-doubly stochastic if

$$\int_X Tf\, d\mu = \int_X f\, d\mu, \quad \forall f \in L^1,$$

$$\int_X T^* \chi_E\, d\mu \leq \mu(E) \ \ \forall E \in \mathcal{A} \ \ with \ \ \mu(E) < \infty$$

The set of all semi-doubly stochastic operators on $L^1(X, \mu)$ is denoted by $s\mathcal{DS}(L^1))$.
It is clear that $\mathcal{DS}(L^1(X, \mu)) \subset S\mathcal{DS}(L^1(X, \mu))$.But the converse is not true in general.

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Main Results

Theorem 10

Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Then

$$\mathcal{DS}(L^1(X,\mu)) = S\mathcal{DS}(L^1(X,\mu)).$$

Main Results

Theorem 11

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $T : L^1 \to L^1$ be a positive bounded linear operator. Then the following are equivalent:

(1) *For each $f \in L^1$, $Tf \prec f$.*

(2) *T is semi doubly stochastic operators on L^1.*

Main Results

For $f \in L^1(X,\ \mu)$, let $S_f := \{Sf;\ S \in \mathcal{SDS}(L^1)\}$ and $\Omega_f := \{h \in L^1; h \geq 0 \text{ and } h \prec f\}$.

Theorem 12

Let (X, μ) be a σ-finite measure space. For $f \in L^1$, the set S_f is dense in Ω_f .

Hence the majorization relation on $L^1(X, \mu)$, for a σ-finite measure space, can be characterized as follows.

Corollary 13

Let X be a σ-finite measure space. Then for $f, g \in L^1(X, \mu)$,

$$g \prec f \Leftrightarrow \exists (S_n)_{n\in\mathbb{N}} \in \mathcal{SDS}\big(L^1(X,\mu)\big) \ \ s.t \ \ S_n f \xrightarrow{L^1} g.$$

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For $f \in L^1(X, \mu)$, let $S_f := \{Sf;\ S \in S\mathcal{D}S(L^1)\}$ and $\Omega_f := \{h \in L^1; h \geq 0 \text{ and } h \prec f\}$.

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Extension of “only if” part of Nielsen’s theorem

Theorem 14

If for some $S \in s\mathcal{DS}(l^1)$, $\lambda_\psi = S\lambda_\phi$, then $|\psi\rangle$ is convertible to $|\phi\rangle$ by LOCC.

If

$$|\phi\rangle = \sum_{i=1}^{\infty} \sqrt{(\lambda_\phi)_i}\; e_i \otimes f_i$$

is target pure state, we can identify

$$\mathfrak{A} = \left\{ |\psi\rangle = \sum_{i=1}^{\infty} \sqrt{(S\lambda_\phi)_i}\; e_i^* \otimes f_i^* : \quad S \in \mathcal{SD}(l^1) \right\},$$

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$$|\psi\rangle \hookrightarrow |\phi\rangle$$

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Thank You
For Your Attention