Colorings of Quantum Graphs, Correlations and Operator Algebras

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Motivation for Today's Talk

- In recent years, we have learned that the theory of **non-local games** provides ways of constructing interesting examples of operator algebras.
- The work of [Ji-Natarajan-Vidick-Wright-Yuen] on MIP* = RE provides a synchronous non-local game whose game algebra is not Connes Embeddable.
- However, the construction from MIP* = RE is highly non-explicit and poorly understood.
- In previous talks, we have seen that non-local games with **quantum inputs/outputs** show promise to provide new examples of operator algebras in the quest for an explicit counter-example to Connes.
- This talk is about quantum graphs and the quantum homomorphism games associated to them.
- These examples provide a natural motivation for the study of quantum games, and give interesting examples of quantum correlations and related operator algebras.

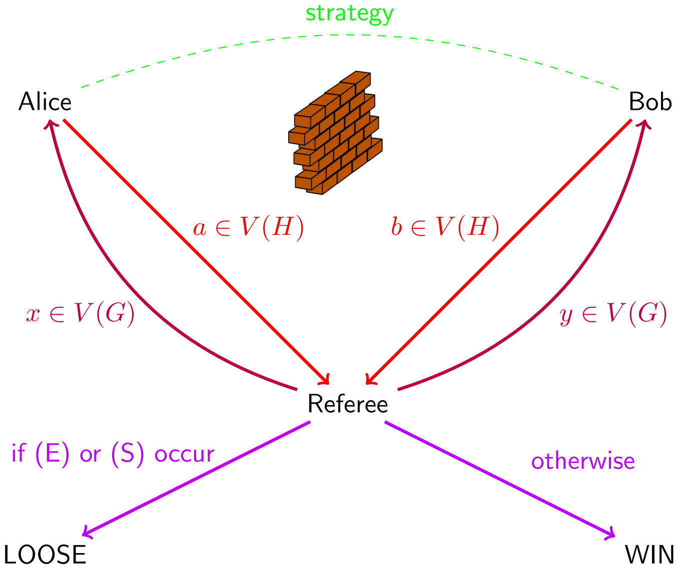
The Graph Homomorphism Game, Hom(G, H)

Let G, H be finite simple graphs. The graph homomorphism game Hom(G, H) is the non-local game given by

- **Inputs**: $X = Y = V(G)$.
- **Oupututs**: $A = B = V(H)$.
- **Rule Function λ**: If, in a given round, (Alice, Bob) receive $(x, y) \in V(G) \times V(G)$ and reply with $(a, b) \in V(H) \times V(H)$, then $\lambda(x, y, a, b) = 1$ unless
  - (E) $(x, y) \in E(G)$ and $(a, b) \notin E(H)$ (Edge Preservation Requirement), or
  - (S) $x = y$ and $a \neq b$ (Synchronicity Requirement)
- **Non-locality**: Alice and Bob devise a strategy to play cooperatively to win each round. But they are unable to communicate while the game is played.
- Alice and Bob are trying to convince the referee that there is a graph homomorphism $G \to H$.

Note: When $H = K_c$, the **complete graph with c vertices**, this is the c-coloring game for G.

Schematic for a round of Hom(G, H)



Winning Strategies for Hom(G, H)

- In general, Alice and Bob's strategy is described by a bipartite probabilistic correlation $p = (p(a, b|x, y))_{x,y \in V(G),\ a,b \in V(H)}$.
- A strategy $p = (p(a, b|x, y))$ is called winning (or perfect) if

$$p(a, b|x, y) = 0 \text{ whenever } \lambda(x, y, a, b) = 0.$$

- Here we will only consider "physically realizable" winning strategies, which are split into two main classes: **classical (or local) strategies** and **quantum strategies**.
- Classical strategies are probabilistic mixtures of deterministic strategies. Quantum strategies are more general, and are achieved by Alice and Bob performing local measurements on a shared quantum state.
- Since Hom(G, H) is **synchronous**, we can completely characterize the existence of a winning strategy (classical/quantum) in terms of operator algebras!

Winning Strategies for $\mathrm{Hom}(G, H)$ via operator algebras

Let G, H be graphs. Define $\mathcal{A}(\mathrm{Hom}(G, H))$ to be the universal unital $*$-algebra with generators $(p_{xa})_{x \in V(G), a \in V(H)}$ subject to the relations

1. For each $x \in V(G)$, $(p_{xa})_{a \in V(H)}$ is a PVM:

$$p_{xa} = p_{xa}^* = p_{xa}^2 \quad \& \quad \sum_a p_{xa} = 1.$$

2. For each $(x, y) \in E(G)$ and $(a, b) \notin E(H)$,

$$p_{xa} p_{yb} = 0.$$

Theorem (Ortiz-Paulsen,Helton-Meyer-Paulsen-Satriano)

There exists a winning (quantum commuting) strategy for Hom(G, H) (written $G \to_{qc} H$) if and only if there exists a non-zero tracial C^*-algebra (B, τ) and a unital $*$-homomorphism $\pi : \mathcal{A}(Hom(G, H)) \to B$. The associated winning correlation is given by $p(a, b|x, y) = \tau \circ \pi(p_{xa} p_{yb})$.

LOC, Q, and QA Strategies

Theorem (Ortiz-Paulsen,Helton-Meyer-Paulsen-Satriano)

$G \to_{qc} H$ if and only if $\mathcal{A}(Hom(G,H))$ has a non-zero representation in some tracial C-algebra (B,τ).*

We say that the game Hom(G,H) has a

- winning Local Strategy $(G \to_{loc} H)$ if we can take B to be abelian.
- winning Quantum Strategy $(G \to_q H)$ if we can take B to be finite dimensional.
- winning Quantum Approximate Strategy $(G \to_{qa} H)$ if we can take $B = R^\omega$.

Intuition: $\mathcal{A}(\text{Hom}(G,H))$ is a free analogue of the algebra of coordinate functions on the set of graph homomorphisms $G \to H$.
Note that

$$(G \to H) \iff (\to_{loc}) \implies (\to_q) \implies (\to_{qa}) \implies (\to_{qc}).$$

Quantum Chromatic and Independence Numbers

For $t \in \{loc, q, qa, qc\}$, can define quantum analogues of analogues chromatic numbers and independence numbers $\chi(G)$ and $\alpha(G)$:

- The t-chromatic number of a graph G is

$$\chi_t(G) = \min_c \{\exists G \to_t K_c\}.$$

- The t-independence number of a graph G is

$$\alpha_t(G) = \max_c \{\exists K_c \to_t \overline{G}\}.$$

- $\chi(G) = \chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G)$,
  $\alpha(G) = \alpha_{loc}(G) \leq \alpha_q(G) \leq \alpha_{qa}(G) \leq \alpha_{qc}(G)$.

Theorem (Mancinska-Roberson-Varvitsiotis)

Using MIP=RE, $\exists$ a graph G such that $\alpha_{qc}(G) > \alpha_{qa}(G)$.*

Good news: The algebras $\mathcal{A}(\text{Hom}(G, H))$ are rich enough to witness the failure of Connes!

Bad news: The M-R-V graph G is non-explicit, and likely enormous.

Quantum Graphs and Quantum Games

- We'd really like a tractable and explicit example of a game algebra $\mathcal{A}(\mathsf{Hom}(G, H))$ that witnesses the failure of the Connes embedding property.
- One approach (already suggested in Ivan Todorov's talk) is to replace classical input/output games by more general quantum input/quantum output games.
- I want to promote this idea by discussing quantum versions of the graph homomorphism game $\mathsf{Hom}(G, H)$, involving **quantum graphs**.

From Graphs to Quantum Graphs

Given a graph G, we can associate the following algebraic data:

- The finite dimensional C*-algebra $M = C(V(G))$
- A Hilbert space $L^2(M) = \ell^2(V(G))$, on which $M = M'$ is represented as the **diagonal algebra**.
- A **self-adjoint subspace**

$$S = S_G = \text{span}\{e_{xy} : (x, y) \in E(G)\} \subset B(L^2(M)).$$

The pair $(S, M \subset B(L^2(M)))$ has the following properties:

- $S_G \subset B(L^2(M))$ is an $M' - M'$**-bimodule**.
- $S_G \subseteq (M')^{\perp}$.
- G can be recovered from S_G because

$$(x, y) \in E(G) \iff \exists e_{xx} T e_{yy} \neq 0 \text{ for some } T \in S_G$$

Theorem (Weaver)

Let V be a finite set and $M = C(V) \subset B(\ell^2(V))$. There is a 1-1 correspondence between (1) Self-adjoint $M' - M'$-bimodules $S \subseteq (M')^{\perp} \subset B(\ell^2(V))$ and (2) graph structures $G = (V, E)$.

Quantum Graphs

Definition (Weaver, Duan-Severini-Winter, Stahlke)

Let $M \subset B(L^2(M))$ be a (standardly represented) finite-dimensional C*-algebra. A (simple, undirected) Quantum Graph on M is a pair $\mathbb{G} = (S, M)$ where

- $S = S^* \subseteq B(L^2(M))$ is an $M' - M'$-bimodule, and
- $S \subseteq (M')^\perp$.

Intuition: $M =$ "$C(V)$", algebra of functions on a **quantum vertex set** V. $S \subset B(L^2(M))$ tells us which quantum vertices are **adjecent**. The condition $S \perp M'$ is a non-commutative analogue of "**no self-loops**".

Basic Quantum Examples:

- Trivial quantum graph: $S = \{0\} \subset B(L^2(M))$.
- Complete quantum graph: $S = (M')^\perp \subset B(L^2(M))$.
- Quantum graphs on $M_2 \cong$ span{some Pauli matr. $\sigma_x, \sigma_y, \sigma_z$}

Quantum Graph Homomorphism Game, Hom$(\mathbb{G}_1, \mathbb{G}_2)$

Let $\mathbb{G}_i = (S_i, M_i \subseteq B(L^2(M_i)))$ be quantum graphs on M_i $(i = 1, 2)$. We want to define a quantum input-quantum output nonlocal game Hom$(\mathbb{G}_1, \mathbb{G}_2)$, generalizing the classical graph homomorphism game Hom(G, H).

Classically:

- **Inputs**: $= V(G) \times V(G)$, **Outputs**: $= V(H) \times V(H)$.
- Alice and Bob's joint **input-output behavior** is described by a correlation/strategy $p = (p(a, b|x, y))$, or equivalently by a classical noisy bipartite channel

$$\Gamma_p : \ell^1(V(G) \times V(G)) \to \ell^1(V(H) \times V(H));$$
$$\delta_x \otimes \delta_y \mapsto \sum_{a,b} p(a, b|x, y)\delta_a \otimes \delta_b$$

- **Winning Requirement 1**: $\Gamma_p(C(V(G))) \subseteq C(V(H))$.
- **Winning Requirement 2**: $\Gamma_p(S_G) \subseteq S_H$.
  (Identifying $B(\ell^2(V)) = \ell^1(V \times V)$; $e_{xy} \leftrightarrow \delta_x \otimes \delta_y$)

Quantum Graph Homomorphism Game

Let $\mathbb{G}_i = (S_i, M_i \subseteq B(L^2(M_i)))$ be quantum graphs on M_i $(i = 1, 2)$. The quantum game Hom$(\mathbb{G}_1, \mathbb{G}_2)$, is given by

- **Inputs**: Mixed quantum states in $B(L^2(M_1) \otimes \overline{L^2(M_1)})$
- **Outputs**: Mixed quantum states in $B(L^2(M_2) \otimes \overline{L^2(M_2)})$.
- Alice and Bob's joint **input-output behavior**: A no-signalling quantum correlation! I.e., a quantum channel

$$\Gamma : B(L^2(M_1) \otimes \overline{L^2(M_1)}) \to B(L^2(M_2) \otimes \overline{L^2(M_2)});$$
$$(\mathrm{Tr} \otimes \mathrm{id})\Gamma(\rho_1 \otimes \rho_2) = 0 \quad \forall \mathrm{Tr}(\rho_1) = 0$$
$$(\mathrm{id} \otimes \mathrm{Tr})\Gamma(\rho_1 \otimes \rho_2) = 0 \quad \forall \mathrm{Tr}(\rho_2) = 0$$

- With the usual identification $L^2(M_i) \otimes \overline{L^2(M_i)} \cong B(L^2(M_i))$, we can identify $M_i, M_i', S_i \subseteq L^2(M_i) \otimes \overline{L^2(M_i)}$
- **Winning Requirement 1**: $\langle \Gamma(\xi\xi^*) | \eta\eta^* \rangle = 0 \ \forall \xi \in M_1', \ \eta \in (M_2')^\perp$.
- **Winning Requirement 2**: $\langle \Gamma(\xi\xi^*) | \eta\eta^* \rangle = 0 \ \forall \xi \in S_1, \ \eta \in (S_2)^\perp$.

Perfect Strategies and Operator Algebras

A perfect strategy for Hom$(\mathbb{G}_1, \mathbb{G}_2)$ is a channel Γ satisfying requirements 1 and 2 above.

Theorem (Harris-Todorov-Turowska-B, Ganesan-Harris-B, Bochniak-Kasprzak-Soltan)

There exists a unital $*$-algebra $\mathcal{A}(Hom(\mathbb{G}_1, \mathbb{G}_2))$ whose non-zero representations in tracial C*-algebras (B, τ) encode the winning quantum commuting strategies for Hom$(\mathbb{G}_1, \mathbb{G}_2)$.

Special Case: Quantum-to-Classical Homomorphisms

Let $\mathbb{G} = (S, M)$ be any quantum graph and let H be a classical graph.

Theorem (Ganesan-Harris-B)

The game Hom$(\mathbb{G}, H)$ has a perfect qc-strategy if and only if there exists a tracial C-algebra (B, τ) and a unital $*$-homomorphism*

$$\rho : C(V(H)) \to M \otimes B; \quad \rho(e_{aa}) = P_a \qquad (a \in V(H))$$

satisfying

$$P_a(S \otimes 1)P_b = 0 \qquad \forall (a, b) \notin E(H).$$

- The game algebra $\mathcal{A}(\text{Hom}(\mathbb{G}, H))$ is the "universal B" generated by the coefficients of ρ.
- When $\mathbb{G} = G$ is classical, we recover the original game algebra $\mathcal{A}(\text{Hom}(G, H))$.
- Can talk about various classes of quantum homomomorphisms $\mathbb{G} \to_t H$, $t \in \{loc, q, qa, qc\}$.

Quantum Colorings of Complete Quantum Graphs

Here's a simple example showing how potentially interesting these algebras can be:

- Denote by QK_n the complete quantum graph on M_n, with $S = (M_n')^\perp \subset B(L^2(M_n))$.
- **Problem**: What's the t-chromatic number of QK_n? $\chi_t(QK_n) = \min_c\{\exists QK_n \to_t K_c\}$? $t \in \{loc, q\}$

Theorem (Ganesan-Harris-B)

$\chi_{loc}(QK_n) = \infty$ and $\chi_q(QK_n) = n^2$.

Proof: Study representations of the game algebra $\mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$, which has generators $(p_{ij}^{(a)})_{1\le a\le n^2, 1\le i,j\le n}$ with relations

$$(p_{ij}^{(a)})^* = p_{ji}^{(a)}, \quad p_{ij}^{(a)} p_{kl}^{(a)} = \delta_{jk} n^{-1} p_{il}^{(a)}, \quad \sum_j p_{ij}^{(a)} p_{jk}^{(b)} = \delta_{ab} p_{ik}^{(a)},$$

$$\sum_i p_{ii}^{(a)} = n^{-1} 1, \quad \sum_a p_{ij}^{(a)} = \delta_{ij} 1.$$

Small representations of $\mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$

- Can find representations $\pi : \mathcal{A} = \mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2})) \to M_n$ using unitary error bases

$$\mathcal{B} = \{u_a\}_{1 \leq a \leq n^2} \subset \mathcal{U}(M_n) \quad \text{s.t.} \quad \mathrm{tr}(u_a^* u_b) = \delta_{ab}.$$

- Every unitary error basis $\mathcal{B}$ gives a (non-abelian) representation

$$\pi : \mathcal{A} \to M_n; \quad \pi(p_{ij}^{(a)}) = \frac{1}{n}(u_a^* e_{ij} u_a).$$

$\implies \chi_q(QK_n) \leq n^2$.

- $\chi_q(QK_n) \geq n^2$ follows from a quantum Hoffman spectral bound on χ_q (**Ganesan '21**).

Big representations of $\mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$

What else can we say about the algebra $\mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$ and its traces?

- Is $\mathcal{A} = \mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$ residually finite-dimensional?
- Is every trace on $\mathcal{A}$ amenable?

Theorem (Gao-Weeks-B)

If $n \geq 3$, there exists a **faithful trace** τ *on $\mathcal{A}$ whose GNS von Neumann algebra $M = \pi_\tau(\mathcal{A})''$ is a strongly solid, non-injective II$_1$-factor.*

This says that the coloring game algebra $\mathcal{A}(\mathrm{Hom}(QK_n, K_{n^2}))$ is very "wild".

Question

Let Γ_τ be the QNS correlation (winning strategy for $\mathrm{Hom}(QK_n, K_{n^2})$) associated to the faithful trace τ above. Is Γ_τ a q/qa/qc-correlation? Is $M = \pi_\tau(\mathcal{A})''$ Connes embeddable?

Very simple quantum games give very interesting, yet poorly understood operator algebras! **THANKS!**