Bounds for k-diametral point configurations in Minkowski spaces

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Introduction (From Petty to Borsuk and Hadwiger):

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Citations	From References: 38	From Reviews: 6

MR275294 (43 #1051) 52.50
Petty, C. M. [Petty, Clinton M.]
Equilateral sets in Minkowski spaces.
Proc. Amer. Math. Soc. **29** (1971), 369–374.

In this paper a connection between equilateral and antipodal sets in an n-dimensional Minkowski space M^n is established. (A subset of a metric space is called equilateral if every two points of the subset have the same distance between them. A subset S of an n-dimensional real linear space is said to be antipodal if for each $p, g \in S$ there exist disjoint parallel support hyperplanes P, Q such that $p \in P$, $g \in Q$.) A certain characterization of equilateral sets in M^n which are strictly antipodal, and the cardinality of maximal equilateral sets in M^n, are also given. *N. Poláková*

Petty's theorems (1971):

2. **Equilateral sets.** A subset S of an n-dimensional real linear space R^n is said to be antipodal provided for each pair of points $p, q \in S$ there exist disjoint parallel support hyperplanes P, Q such that $p \in P$, $q \in Q$. Properties of antipodal sets have been studied in [**3**] and [**5**]. An antipodal set is finite and S is the set of vertices of the antipodal polytope $A = \text{conv } S$.

A line L in a Minkowski space M^n is normal to a hyperplane H (or H is transversal to L) at a point f if $L \cap H = \{f\}$ and the distance $xf \leqq xy$ for $x \in L$ and $y \in H$. The line through two points p and q will be denoted by $L(p, q)$.

THEOREM 1. *Let S be an equilateral set in a Minkowski space M^n. Then S is an antipodal set such that for every pair of points $p, q \in S$ the hyperplanes through p and q transversal to $L(p, q)$ are support hyperplanes to S.*

Theorem 2. *Let S be an antipodal set in R^n. Then there exists a norm on R^n such that S is an equilateral set.*

An equilateral set is said to be maximal if it is not a proper subset of any other equilateral set.

Theorem 4. *If S is a maximal equilateral set in M^n then*

$$\min(4, n+1) \leqq \operatorname{card} S \leqq 2^n,$$

Danzer-Grunbaum theorem (1962):

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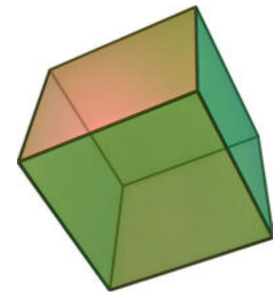

MR138040 (25 #1488) 52.30

Danzer, L. [Danzer, Ludwig W.]; Grünbaum, B. [Grünbaum, Branko]

Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee. (German)

Math. Z. **79** (1962), 95–99.

If V is the set of 2^n vertices of an n-dimensional cube in E^n, then V has the following two properties: (i) no obtuse angle is determined by three points of V; (ii) each two points p and q of V are antipodal in the sense that there are parallel hyperplanes P and Q such that $p \in P$, $q \in Q$, and the entire set V lies between P and Q. The authors prove that if a subset of E^n has either of these two properties, then it includes at most 2^n points. (This settles questions raised by P. Erdős and the reviewer.) This result is sharpened in various ways and some related questions are discussed.

Petty's conjecture (1971):

Let $(X, \|\cdot\|)$ be a normed space. A set $S \subseteq X$ is called *c-equilateral* if $\|x - y\| = c$ for all distinct $x, y \in S$. S is called *equilateral* if it is c-equilateral for some $c > 0$. The *equilateral number* $e(X)$ of X is the cardinality of the largest equilateral set of X. Petty [Pet71] made the following conjecture regarding lower bounds on $e(X)$.

Conjecture 1 (Petty[Pet71]). *For all normed spaces X of dimension n, $e(X) \geq n + 1$.*

Petty [Pet71] proved Conjecture 1 for $n = 3$, and Makeev [Mak05] for $n = 4$. For $n \geq 5$ the conjecture is still open, except for some special classes of norms.

[Mak05] Vladimir Vladimirovich Makeev, *Equilateral simplices in normed 4-space*, Zapiski Nauchnykh Seminarov POMI **329** (2005), 88–91.

For further info see:

[Swa18] K. J. Swanepoel, *Combinatorial distance geometry in normed spaces*, New Trends in Intuitive Geometry, Springer, 2018, pp. 407–458.

Borsuk's conjecture (1933):

Definition 2. *Let* $b(\mathbb{M}^d_{\mathbf{K_o}})$ *denote the smallest positive integer* m *such that any finite set* $Y \subset \mathbb{M}^d_{\mathbf{K_o}}$ *with diameter* $\mathrm{diam}_{\mathbb{M}^d_{\mathbf{K_o}}}(Y) > 0$ *can be partitioned into* m *sets each having diameter smaller than* $\mathrm{diam}_{\mathbb{M}^d_{\mathbf{K_o}}}(Y)$ *in* $\mathbb{M}^d_{\mathbf{K_o}}$. *We call* $b(\mathbb{M}^d_{\mathbf{K_o}})$ *the* Borsuk number *of the Minkowski space* $\mathbb{M}^d_{\mathbf{K_o}}$.

It is well known that Borsuk [6] asked whether $b(\mathbb{E}^d) = d + 1$ for any $d > 2$

[6] K. Borsuk, Drei Sätze über die n-dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

Borsuk's conjecture was proved in dimensions 2 and 3

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, July 1993

A COUNTEREXAMPLE TO BORSUK'S CONJECTURE

JEFF KAHN AND GIL KALAI

ABSTRACT. Let $f(d)$ be the smallest number so that every set in R^d of diameter 1 can be partitioned into $f(d)$ sets of diameter smaller than 1. Borsuk's conjecture was that $f(d) = d + 1$. We prove that $f(d) \geq (1.2)^{\sqrt{d}}$ for large d.

MR3292266 52C17 05E30
Jenrich, Thomas; Brouwer, Andries E. (NL-MATH)
A 64-dimensional counterexample to Borsuk's conjecture. (English summary)
Electron. J. Combin. **21** (2014), *no. 4*, *Paper* 4.29, 3 *pp.*

On the one hand, Lassak [20] has proved that $b(\mathbb{E}^d) \leq 2^{d-1} + 1$ for any $d > 1$.

[20] M. Lassak, An estimate concerning Borsuk partition problem, *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **30/9-10** (1982), 449–451.

On the other hand, Schramm [25] proved the inequality $b(\mathbb{E}^d) \leq 5d\sqrt{d}(4 + \ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}$ for all $d > 1$.

[25] O. Schramm, Illuminating sets of constant width, *Mathematika* **35/2** (1988), 180–189.

For a survey on the Borsuk partition problem see:

[24] A. M. Raigorodskii, Three lectures on the Borsuk partition problem, *London Math. Soc. Lecture Note Ser.* **347** Cambridge Univ. Press., Cambridge, 2008, 202–247.

Hadwiger's conjecture (1957):

Definition 3. *Let* $h(\mathbb{R}^d)$ *denote the smallest positive integer* l *such that the convex hull* $\mathrm{conv}(Y)$ *of any finite set* $\emptyset \neq Y \subset \mathbb{R}^d$ *can be covered by* l *smaller positive homothetic copies, i.e., there exist* $0 < \lambda_i < 1$, $\mathbf{y}_i \in \mathbb{R}^d$ *for* $1 \le i \le l$ *such that* $\mathrm{conv}(Y) \subseteq \bigcup_{i=1}^{l} (\mathbf{y}_i + \lambda_i \mathrm{conv}(Y))$. *We call* $h(\mathbb{R}^d)$ *the* Hadwiger number *of* $\mathbb{R}^d$.

Hadwiger [14] conjectured that if $d > 2$, then $h(\mathbb{R}^d) = 2^d$.

[14] H. Hadwiger, Ungelöstes Probleme Nr. 20, *Elem. Math.* **12/6** (1957), 121.

Rogers [26] (see also [27]) has proved

that $h(\mathbb{R}^d) \le \binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(4^d \sqrt{d} \ln d)$ holds for any $d > 1$.

[26] C. A. Rogers, A note on coverings, *Mathematika* **4** (1957), 1–6.

[27] C. A. Rogers and C. Zong, Covering convex bodies by translates of convex bodies, *Mathematika* **44/1** (1997), 215–218.

Lassak [21] improved this upper bound of Rogers for some small values of d by showing that $h(\mathbb{R}^d) \leq (d+1)d^{d-1} - (d-1)(d-2)^{d-1}$ for any $d > 1$.

[21] M. Lassak, Covering the boundary of a convex set by tiles, *Proc. Am. Math. Soc.* **104** (1988), 269–272.

Finally, just very recently Huang, Slomka, Tkocz, and Vritsiou [16] improved the upper bound of Rogers for sufficiently large values of d by showing that there exist universal constants $c_1, c_2 > 0$ such that for all $d > 1$, one has $h(\mathbb{R}^d) \leq c_1 4^d e^{-c_2\sqrt{d}}$.

[16] H. Huang, B. A. Slomka, T. Tkocz, and B.-H. Vritsiou, Improved bounds for Hadwiger's covering problem via thin-shell estimates, arXiv:1811.12548v1 [math.MG] 30 Nov 2018, 1–19.

For a survey on the Hadwiger covering problem see:

[4] K. Bezdek and M. A. Khan, The geometry of homothetic covering and illumination, *Discrete Geometry and Symmetry, Springer Proc. Math. Stat.* **234** Springer, 2018, 1–30.

New Results:

On k-diametral point configurations in Minkowski spaces *

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arXiv:2005.04542v1 [math.MG] 10 May 2020

Abstract

The structure of k-diametral point configurations in Minkowski d-space is shown to be closely related to the properties of k-antipodal point configurations in $\mathbb{R}^d$. In particular, the maximum size of k-diametral point configurations of Minkowski d-spaces is obtained for given $k \geq 2$ and $d \geq 2$ generalizing Petty's results (Proc. Am. Math. Soc. 29: 369-374, 1971) on equilateral sets in Minkowski spaces. Furthermore, bounds are derived for the maximum size of k-diametral point configurations in Euclidean d-space. In the proofs convexity methods are combined with volumetric estimates and combinatorial properties of diameter graphs.

Definition (k-diametral point configuration):

Let $\mathbf{K_o} \subset \mathbb{R}^d$ be an **o**-symmetric convex body, i.e., a compact convex set with nonempty interior symmetric about the origin **o** in $\mathbb{R}^d$. Let $\mathcal{K}_{\mathbf{o}}^d$ denote the family of **o**-symmetric convex bodies in $\mathbb{R}^d$. Moreover, let $\|\cdot\|_{\mathbf{K_o}}$ denote the norm generated by $\mathbf{K_o} \in \mathcal{K}_{\mathbf{o}}^d$, which is defined by $\|\mathbf{x}\|_{\mathbf{K_o}} := \min\{\lambda \geq 0 \mid \mathbf{x} \in \lambda\mathbf{K_o}\}$ for $\mathbf{x} \in \mathbb{R}^d$. Furthermore, let us denote $\mathbb{R}^d$ with the norm $\|\cdot\|_{\mathbf{K_o}}$ by $\mathbb{M}_{\mathbf{K_o}}^d$ and call it the *Minkowski space of dimension d generated by* $\mathbf{K_o}$. The following definition introduces the central notion for our paper.

Definition 1. *We call the labeled point set* $X := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \subset \mathbb{R}^d$ *a* point configuration of n points *in* $\mathbb{R}^d$, *where* $n \geq 1$ *and* $d \geq 2$. *Here the points* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ *are not necessarily all distinct and therefore n is not necessarily equal to the number of distinct points in X. Next, let* $X := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \subset \mathbb{R}^d$ *be a point configuration of n points with some positive diameter in* $\mathbb{M}_{\mathbf{K_o}}^d$, *i.e., let* $\mathrm{diam}_{\mathbb{M}_{\mathbf{K_o}}^d}(X) := \max\{\|\mathbf{x}_i - \mathbf{x}_j\|_{\mathbf{K_o}} \mid 1 \leq i < j \leq n\} > 0$. *Let* $k \geq 2$ *be an integer. Then we say that* $X := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \subset \mathbb{R}^d$ *is a* k-diametral point configuration of n points in $\mathbb{M}_{\mathbf{K_o}}^d$ *if any k-tuple* $\mathbf{x}_{n_1}, \mathbf{x}_{n_2}, \dots, \mathbf{x}_{n_k}$, $1 \leq n_1 < n_2 < \dots < n_k \leq n$ *chosen from X contains two diametral points, i.e., it contains two points say,* $\mathbf{x}_{n_i}$ *and* $\mathbf{x}_{n_j}$, $1 \leq i < j \leq k$ *such that* $\|\mathbf{x}_{n_i} - \mathbf{x}_{n_j}\|_{\mathbf{K_o}} = \mathrm{diam}_{\mathbb{M}_{\mathbf{K_o}}^d}(X)$. *In particular, a 2-diametral point configuration is called a* diametral *or simply an* equilateral *point configuration, and a 3-diametral point configuration is called an* almost diametral *point configuration. Finally, let us denote the largest n for which there exists a k-diametral point configuration of n points in* $\mathbb{M}_{\mathbf{K_o}}^d$, *by* $f_k(\mathbb{M}_{\mathbf{K_o}}^d)$ *and call it the* k-diametral number *of point configurations in* $\mathbb{M}_{\mathbf{K_o}}^d$.

Proposition 3.

(i) *For $d \geq 2$, $k \geq 2$, and $\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}$, one has*

$$f_2(\mathbb{M}^d_{\mathbf{K_o}}) \leq \frac{1}{k-1} f_k(\mathbb{M}^d_{\mathbf{K_o}}) \leq b(\mathbb{M}^d_{\mathbf{K_o}}) \leq h(\mathbb{R}^d) \tag{1}$$

$$\leq \min\left\{\binom{2d}{d} d(\ln d + \ln\ln d + 5), (d+1)d^{d-1} - (d-1)(d-2)^{d-1}, c_1 4^d e^{-c_2\sqrt{d}}\right\}. \tag{2}$$

(ii) $f_k(\mathbb{M}^2_{\mathbf{K_o}}) = (k-1) f_2(\mathbb{M}^2_{\mathbf{K_o}})$ *holds for all $k \geq 2$ and $\mathbf{K_o} \in \mathcal{K}^2_{\mathbf{o}}$.*

(iii) $f_k(\mathbb{E}^d) = (k-1)(d+1)$ *holds for all $k \geq 3$ and $d = 2, 3$ (and for $k = 2$ and all $d \geq 1$). Moreover, if $k \geq 3$ and $d \geq 4$, then*

$$(k-1)(d+1) \leq f_k(\mathbb{E}^d) \leq (k-1) b(\mathbb{E}^d) \tag{3}$$

$$\leq (k-1)\min\left\{2^{d-1} + 1, 5d\sqrt{d}(4 + \ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}\right\}. \tag{4}$$

Proof of Part (ii). Clearly, the inequalities $f_2(\mathbb{M}^2_{\mathbf{K_o}}) \leq \frac{1}{k-1} f_k(\mathbb{M}^2_{\mathbf{K_o}}) \leq b(\mathbb{M}^2_{\mathbf{K_o}})$ (see Part **(i)** of Proposition 3) combined with the following claim complete the proof of Part **(ii)**.

Sublemma 1. *$f_2(\mathbb{M}^2_{\mathbf{K_o}}) = b(\mathbb{M}^2_{\mathbf{K_o}})$ holds for all $\mathbf{K_o} \in \mathcal{K}^2_{\mathbf{o}}$.*

Proof. On the one hand, recall that according to [5] and [13] $b(\mathbb{M}^2_{\mathbf{K_o}}) = 3$ if $\mathbf{K_o} \in \mathcal{K}^2_{\mathbf{o}}$ is different from a parallelogram and $b(\mathbb{M}^2_{\mathbf{K_o}}) = 4$ if $\mathbf{K_o}$ is a parallelogram. On the other hand, Petty [23] proved that $f_2(\mathbb{M}^2_{\mathbf{K_o}}) = 3$ if $\mathbf{K_o} \in \mathcal{K}^2_{\mathbf{o}}$ is different from a parallelogram and $f_2(\mathbb{M}^2_{\mathbf{K_o}}) = 4$ if $\mathbf{K_o}$ is a parallelogram. Thus, Sublemma 1 follows. □

[5] V. G. Boltyanskii and V. P. Soltan, Borsuk's problem, *Mat. Zametki* **22/5** (1977), 621–631.

[13] B. Grünbaum, Borsuk's partition conjecture in Minkowski planes, *Bull. Res. Council Israel Sect. F* **7F** (1957/58), 25–30.

[23] C. M. Petty, Equilateral sets in Minkowski spaces, *Proc. Amer. Math. Soc.* **29** (1971), 369–374.

Proof of Part (iii). Let us start by recalling that $b(\mathbb{E}^2) = 3$ ([6]) and $b(\mathbb{E}^3) = 4$ ([10], [12], and [15]). Combining these facts with $f_2(\mathbb{E}^d) \leq \frac{1}{k-1} f_k(\mathbb{E}^d) \leq b(\mathbb{E}^d)$ (see Part **(i)** of Proposition 3) and the elementary observations that $f_2(\mathbb{E}^2) = 3$ and $f_2(\mathbb{E}^3) = 4$, one obtains that $f_k(\mathbb{E}^d) = (k-1)(d+1)$ holds for all $k \geq 2$ and $d = 2, 3$. In order to prove (3), we observe that the inequalities $(k-1)(d+1) \leq f_k(\mathbb{E}^d) \leq (k-1)b(\mathbb{E}^d)$ follow from Part **(i)** of Proposition 3 and the elementary observation that $f_2(\mathbb{E}^d) = d+1$. Finally, (4) follows from (3) and the results of Lassak [20] and Schramm [28] in a straightforward way. This completes the proof of Proposition 3.

[6] K. Borsuk, Drei Sätze über die n-dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

[10] H. G. Eggleston, Covering a three-dimensional set with sets of smaller diameter, *J. London Math. Soc.* **30** (1955), 11–24.

[12] B. Grünbaum, A simple proof of Borsuk's conjecture in three dimensions, *Proc. Cambridge Philos. Soc.* **53** (1957), 776–778.

[15] A. Heppes, On the partitioning of a three-dimensional point set into sets of smaller diameter (in Hungarian), *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **7** (1957), 413–416.

Definition (k-antipodal point configuration):

Definition 4. *Let* $X := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subset \mathbb{R}^d$ *be a point configuration of* n *points in* $\mathbb{R}^d$, *where* $n \geq 1$ *and* $d \geq 2$. *Let* $k \geq 2$ *be an integer. We say that* X *is a* k-antipodal point configuration *in* $\mathbb{R}^d$ *if any* k*-tuple* $\mathbf{x}_{n_1}, \mathbf{x}_{n_2}, \ldots, \mathbf{x}_{n_k}$, $1 \leq n_1 < n_2 < \cdots < n_k \leq n$ *chosen from* X *contains two antipodal points, i.e., it contains two points say,* $\mathbf{x}_{n_i}$ *and* $\mathbf{x}_{n_j}$, $1 \leq i < j \leq k$ *lying on distinct parallel supporting hyperplanes of the convex hull* $\mathrm{conv}(X)$ *of* X *in* $\mathbb{R}^d$. *In particular, a 2-antipodal point configuration is called an* antipodal *point configuration, and a 3-antipodal point configuration is called an* almost antipodal *point configuration. Finally, let us denote the largest* n *for which there exists a* k*-antipodal point configuration of* n *points in* $\mathbb{R}^d$, *by* $F_k(d)$ *and call it the* k-antipodal number *of point configurations in* $\mathbb{R}^d$.

Remark 7. *Recall that according to Danzer and Grünbaum [8],* $F_2(d) = 2^d$ *for all* $d \geq 2$. *Furthermore, their volumetric method combined with an earlier result of Groemer [11] (on tiling a convex body into homothetic convex bodies) implies that if* X *is an antipodal point configuration of* 2^d *points in* $\mathbb{R}^d$, *then* X *must be identical to the vertex set of an affine d-cube.*

[8] L. Danzer and B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee, *Math. Zeitschrift* **79** (1962), 95–99.

[11] H. Groemer, Abschätzungen für dia Anzahl der konvexen Körper, die einen konvexen Körper berühren, *Monatsh. Math.* **65** (1961), 74–81.

Hilfssatz 2: Ein konvexer Körper M kann nur dann in zu M homothetische Körper zerlegt werden, wenn er ein Parallelepiped ist.

A generalization of Petty's theorem:

Theorem 8. *The point configuration X is a k-antipodal point configuration in $\mathbb{R}^d$ if and only if there exists $\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}$ such that X is a k-diametral point configuration in $\mathbb{M}^d_{\mathbf{K_o}}$, where $d \geq 2$ and $k \geq 2$. Moreover,*

$$F_k(d) = \max_{\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}} f_k(\mathbb{M}^d_{\mathbf{K_o}}) = (k-1)2^d \tag{5}$$

holds for all $d \geq 2$ and $k \geq 2$. Furthermore, $f_k(\mathbb{M}^d_{\mathbf{K_o}}) = (k-1)2^d$ if and only if $\mathbf{K_o}$ is an $\mathbf{o}$-symmetric affine d-cube of $\mathbb{R}^d$ in which case every k-diametral point configuration of $(k-1)2^d$ points is identical to the vertex set of a homothetic affine d-cube with each vertex having multiplicity $k-1$.

In the proof of the above theorem the following Groemer-type lemma plays a role:

Definition 5. *We say that the convex d-polytopes $\mathbf{P}_l \subset \mathbb{R}^d$, $1 \leq l \leq N$ form a $(k-1)$-*tiling *of the convex d-polytope $\mathbf{P} \subset \mathbb{R}^d$ if $\mathbf{P}_l \subset \mathbf{P}$ holds for all $1 \leq l \leq N$ and every point of $\mathbf{P}$ which is not a boundary point of any $\mathbf{P}_l, 1 \leq l \leq N$ belongs to the interior of exactly $k-1$ convex d-polytopes chosen from $\mathbf{P}_l, 1 \leq l \leq N$.*

Lemma 12. *Let $\mathbf{P} := \operatorname{conv}(X) \subset \mathbb{R}^d$, $d \geq 1$ be a convex d-polytope and $X := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a point configuration of $N = (k-1)2^d$ points in $\mathbb{R}^d$ with $k \geq 2$ such that the convex d-polytopes $\mathbf{P}_l := \mathbf{x}_l + \frac{1}{2}(\mathbf{P} - \mathbf{x}_l)$, $1 \leq l \leq N$, form a $(k-1)$-tiling in $\mathbf{P}$. Then $\mathbf{P}$ is an affine d-cube and X is its vertex set with each vertex having multiplicity $k-1$.*

On k-antipodal (resp., k-diametral) point sets:

From the point of view of geometry, it is natural to complete this section with k-diametral (resp., k-antipodal) properties of *point sets*, i.e., point configurations consisting of distinct points. By restricting Definition 1 (resp., Definition 4) to point sets, let us denote the largest cardinality of k-diametral (resp., k-antipodal) point sets in $\mathbb{M}^d_{\mathbf{K_o}}$ (resp., $\mathbb{R}^d$), by $g_k(\mathbb{M}^d_{\mathbf{K_o}})$ (resp., $G_k(d)$) and call it the k-diametral number of point sets (resp., k-antipodal number of point sets) in $\mathbb{M}^d_{\mathbf{K_o}}$ (resp., $\mathbb{R}^d$). Clearly, $g_k(\mathbb{M}^d_{\mathbf{K_o}}) \leq f_k(\mathbb{M}^d_{\mathbf{K_o}})$ holds for all $k \geq 2$, $d \geq 2$ and $\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}$ and so, the upper bounds already stated for $f_k(\mathbb{M}^d_{\mathbf{K_o}})$ apply to $g_k(\mathbb{M}^d_{\mathbf{K_o}})$ as well. Also, it is obvious that $G_k(d) \leq F_k(d) = (k-1)2^d$ holds for all $k \geq 3$, $d \geq 2$ with $G_2(d) = F_2(d) = 2^d$ for all $d \geq 2$. Furthermore, we have

Theorem 9.

(i) *$G_k(2) = 2k$ for all $k \geq 2$. Furthermore, $S \subset \mathbb{R}^2$ is a k-antipodal point set with* $\mathrm{card}(S) = 2k$, *if and only if $\mathbf{P} := \mathrm{conv}(S)$ is a $(2s)$-gon for some $s \leq k$ with $S \subset \mathrm{bd}(\mathbf{P})$ such that each side of $\mathbf{P}$ is parallel to another side of $\mathbf{P}$ with both of them containing the same number of points from S.*

(ii) *The point set X is a k-antipodal point set in $\mathbb{R}^d$ if and only if there exists $\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}$ such that X is a k-diametral point set in $\mathbb{M}^d_{\mathbf{K_o}}$ implying that $G_k(d) = \max_{\mathbf{K_o} \in \mathcal{K}^d_{\mathbf{o}}} g_k(\mathbb{M}^d_{\mathbf{K_o}})$, where $d \geq 2$ and $k \geq 2$. Moreover, for all $d \geq 3$ and $k \geq 3$ one has*

$$k \cdot 2^{d-1} \leq G_k(d) \leq (k-1)2^d - 1. \tag{6}$$

Proof of Part (ii). The first claim and the upper estimate of (6) follow from the proof of Theorem 8 in a straightforward way. So, we are left to prove the lower bound of (6). Let us start with the $(2s)$-gon $\mathbf{P} := \mathrm{conv}(S) \subset \mathbb{E}^2$ for some $s \le k$ and $\mathrm{card}(S) = 2k$ having the property that $S \subset \mathrm{bd}(\mathbf{P})$ and each side of $\mathbf{P}$ is parallel to another side of $\mathbf{P}$ with both of them containing the same number of points from S. Then let $\mathbb{E}^2 \subset \mathbb{E}^3$ be a plane through the origin of $\mathbb{E}^3$ and let $\mathbf{P}' := \mathbf{x} + \mathbf{P}$ for some $\mathbf{x} \in \mathbb{E}^3 \setminus \mathbb{E}^2$. Clearly, $\mathrm{conv}(\mathbf{P} \cup \mathbf{P}')$ is a 1-fold prism in $\mathbb{E}^3$ and Part (i) implies that $S \cup (\mathbf{x} + S)$ is a k-antipodal point set of cardinality $2(2k)$ in $\mathbb{E}^3$ lying on the edges of $\mathrm{conv}(\mathbf{P} \cup \mathbf{P}')$. Repeating this process $(d-2)$-times one obtains a $(d-2)$-fold prism in $\mathbb{E}^d$ such that its 1-skeleton contains a k-antipodal point set of cardinality $2^{d-1} \cdot k$ in $\mathbb{E}^d$, finishing the proof of Part (ii).

Open problem:

Part (ii) of Theorem 9 supports

Problem 11. *Prove or disprove that* $G_k(d) = k \cdot 2^{d-1}$ *holds for all* $d \ge 3$ *and* $k \ge 3$.

Theorem 9.

(iii) *If* $\mathbf{C_o}$ *is an* $\mathbf{o}$*-symmetric affine d-cube of* $\mathbb{R}^d$*, then* $g_k(\mathbb{M}^d_{\mathbf{C_o}}) = k \cdot 2^{d-1}$ *holds for all* $d \geq 2$ *and* $k \geq 2$*.*

(iv) *For any k-diametral point set* $S \subset \mathbb{E}^2$*, we have* $\mathrm{card}(S) \leq 2k - 1$*, with equality if and only if S is the vertex set of a regular* $(2k-1)$*-gon. Thus,* $g_k(\mathbb{E}^2) = 2k - 1$ *for all* $k \geq 2$*.*

(v) $2k \leq g_k(\mathbb{E}^3) \leq 3k - 2$ *holds for all* $k \geq 4$*. Furthermore,* $g_3(\mathbb{E}^3) = 6$*, and if* $S \subset \mathbb{E}^3$ *is a 3-diametral point set with* $\mathrm{card}(S) = 6$*, then the diameter graph of S is isomorphic to one of (1-a), ..., (2-e) in Figure 1.*

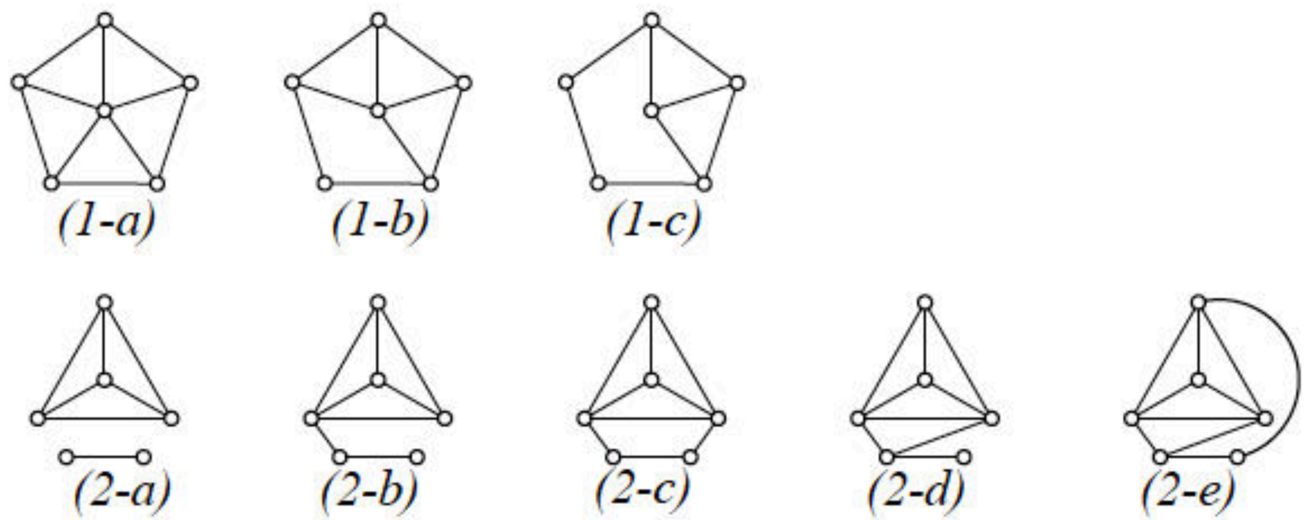

Figure 1: The diameter graphs of 3-diametral point sets in $\mathbb{E}^3$ of maximal cardinality.