Semidefinite programming bounds for the average kissing number

Alexander Kolpakov
(Université de Neuchâtel)

joint work with Maria Dostert (KTH Stockholm) and Fernando Oliveira (TU Delft)

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Average kissing number

Packing $\mathscr{P}$ of balls in $\mathbb{R}^n$: a finite set of closed balls (not necessarily congruent) with disjoint interiors.

Contact graph $G(\mathscr{P})$ of packing $\mathscr{P}$ has vertex set = balls $X, Y, \ldots$; edges defined by X connected to Y iff $X \cap Y \neq \emptyset$.

Average kissing number

Average kissing number in $\mathbb{R}^n$:

$$\kappa(n) = \sup \left\{ \frac{2\,e(G)}{v(G)} \mid G = \text{ contact graph of a packing in } \mathbb{R}^n \right\}$$

$$= \sup\{ \text{ average degree of all contact graphs } \}$$

Average kissing number

- Material science (Torquato et al)
- Spherical codes (Conway—Sloane, ...)
- Extremal problems in lattice theory / geometry

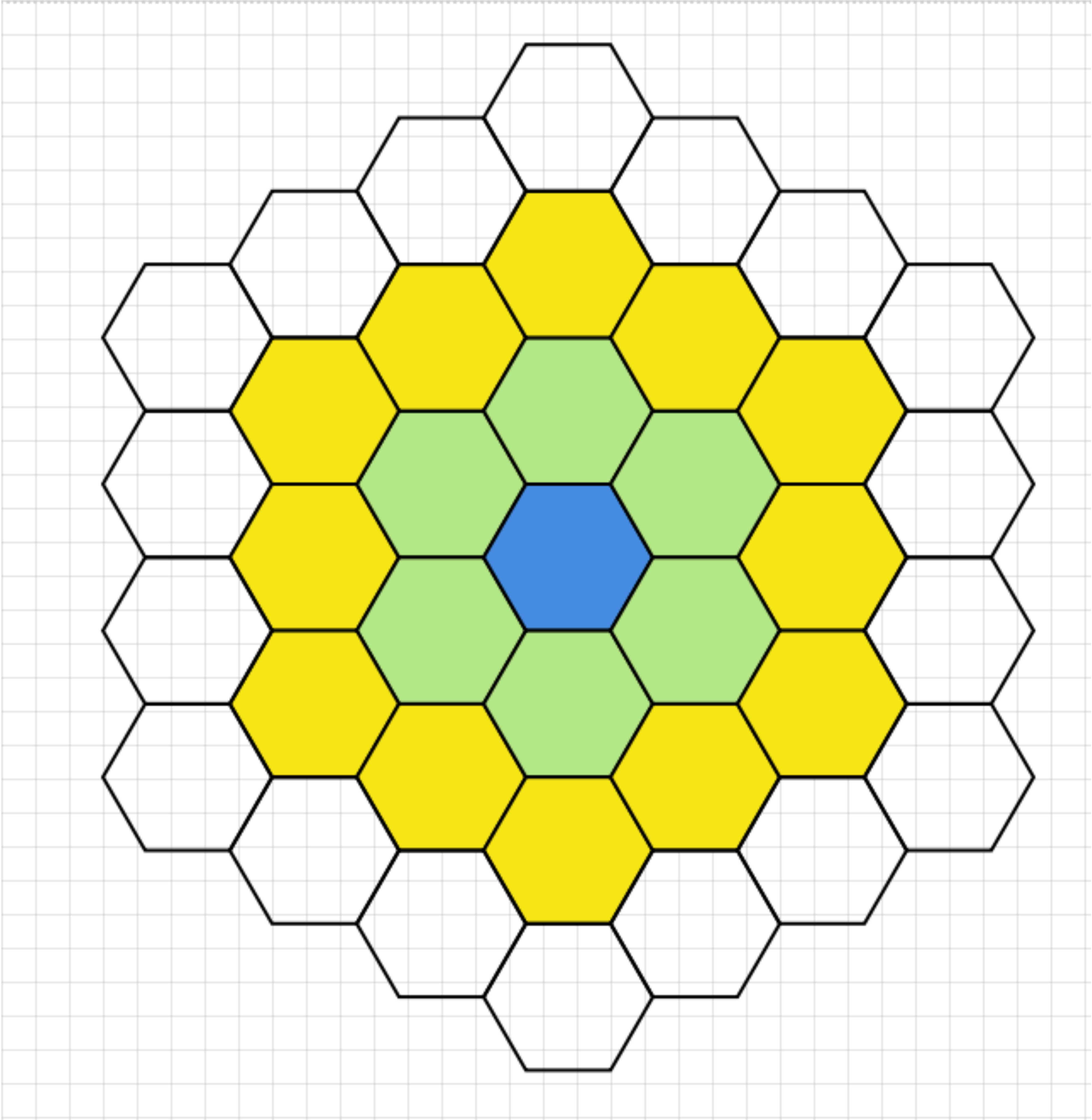
Average kissing number

Köbe–Andreev–Thurston theorem: Contact graphs of disk packings in the plane are simple planar graphs

$$\text{This} \implies \kappa(2) \leq 6$$

Average kissing number

Then $\kappa(2) = 6$ is realised by taking bigger and bigger chunks of the standard hexagonal lattice.



Upper bounds

$K(n) =$ kissing number for congruent radius 1 balls in $\mathbb{R}^n$

Obvious: $K(n)$ is finite

Not obvious: $\kappa(n)$ is finite

Kuperberg & Schramm (1994): $\kappa(n) \leq 2\,K(n)$

Upper bounds

Can we improve over $\kappa(n) \leq 2\,K(n)$?

Kuperberg & Schramm (1994):

$$\kappa(3) \leq 8 + 4\sqrt{3} \approx 14.928\ldots$$

Glazyrin (2017):

$$\kappa(3) < 13.955,\ \kappa(4) < 34.681,\ \kappa(5) < 77.757$$

Upper bounds

Can we improve over $\kappa(n) \leq 2\,K(n)$?

M. Dostert, A.K., F.M. de Oliveira Filho (2020):
improvements in dimensions from 3 to 9 by using semidefinite programming (arXiv:2003.11832)

Lower bounds

Eppstein–Kuperberg–Ziegler (2002):

$$\kappa(3) \geq \frac{666}{53} \approx 12.56603\ldots$$

Lower bounds in other dimensions can be obtained from lattice ball packings (Conway & Sloane: SPLAG)

Best bounds so far

+ comparison

Dimension	*Lower bound*	*Previous upper bound*	*New upper bound*
3	12.612	13.955	13.606
4	24	34.681	27.439
5	40	77.757	64.022
6	72	156	121.105
7	126	268	223.144
8	240	480	408.386
9	272	726	722.629

Best bounds so far

Lower bound in dimension 3: due to Eppstein–Kuperberg–Ziegler.

Other lower bounds: Conway–Sloane [SPLAG, Table 1.2].

Previous upper bounds in dimensions $3, \ldots, 5$: Glazyrin.

Other previous upper bounds: twice the best known upper bound for the kissing number of congruent balls (Kuperberg–Schramm).

Glazyrin's upper bound

Theorem (Glazyrin, 2017): If $n \geq 3$ and $1 < \rho < 3$, then

$$\kappa(n) \leq \frac{\mathrm{dens}(n, \rho)}{A_{n,\rho}(1)},$$

where $A_{n,\rho}(r)$ is the normalised area of the spherical cap in the following configuration:

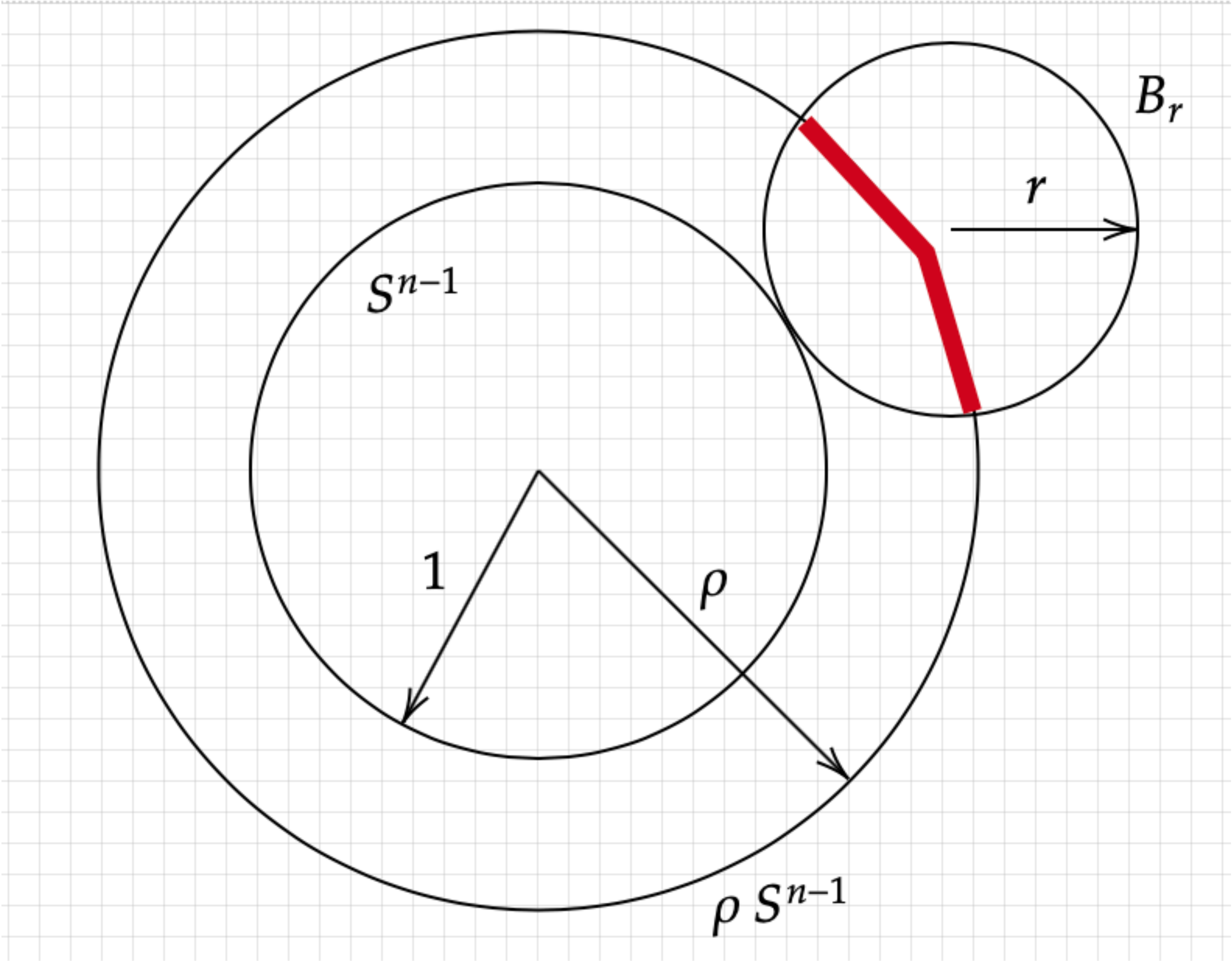
Glazyrin's upper bound

$$A_{n,\rho}(r) = \frac{Area(\rho S^{n-1} \cap B_r)}{Area(\rho S^{n-1})},$$

where S^{n-1} is the unit sphere,

ρS^{n-1} is its ρ—dilate,

and B_r is radius r ball.



Glazyrin's upper bound

... and $\text{dens}(n, \rho)$ is the density function defined as follows: fix $1 < \rho < 3$ and consider a unit ball at origin with boundary = unit sphere S^{n-1}.

Glazyrin's upper bound

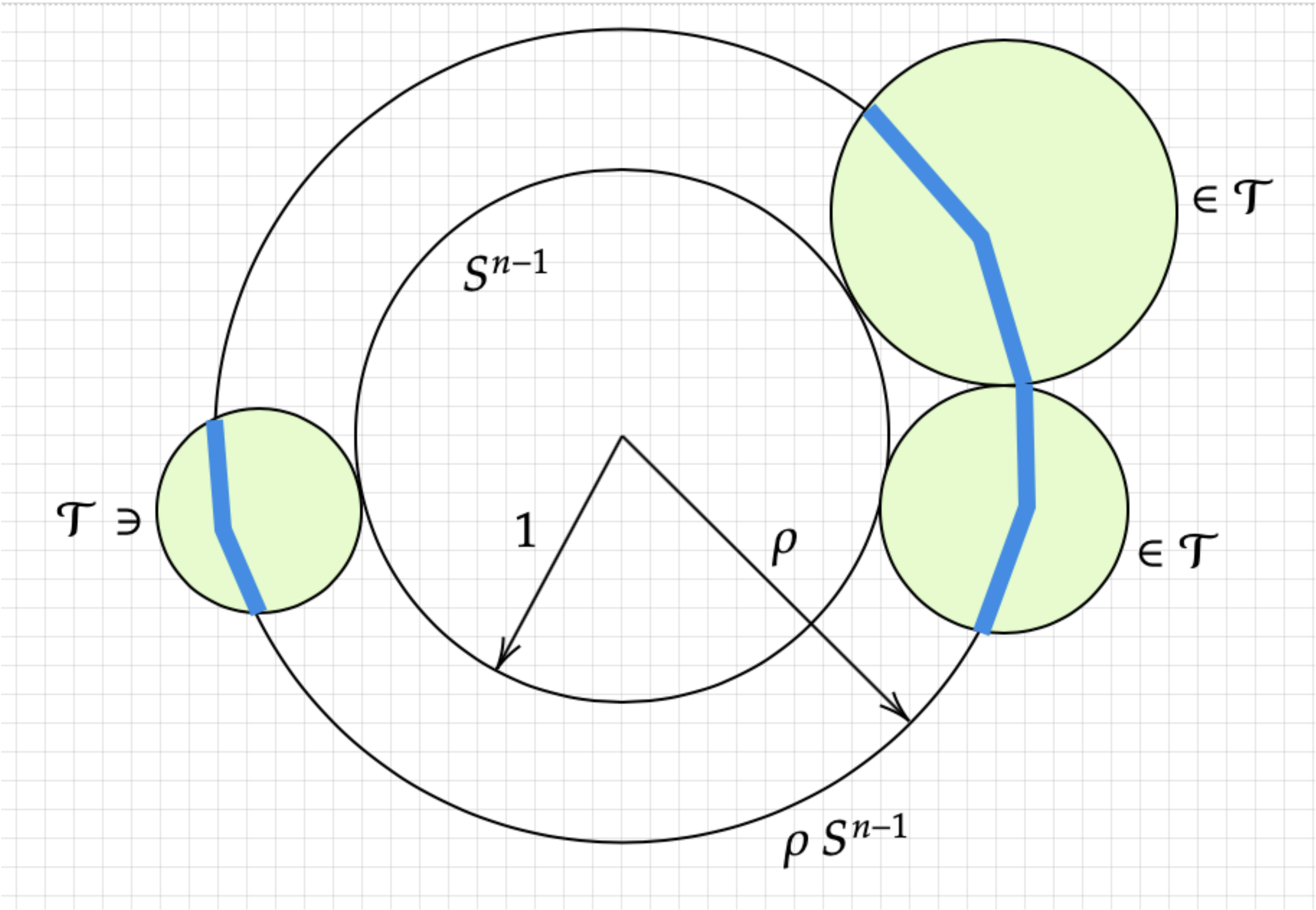
Let $\mathcal{T}$ be any configuration of balls with disjoint interiors tangent to the central unit ball. They cover some fraction $\mathrm{dens}(\mathcal{T}, \rho)$ of the rescaled sphere ρS^{n-1}.

Define: $\mathrm{dens}(n, \rho) = \sup_{\mathcal{T}} \mathrm{dens}(\mathcal{T}, \rho)$

Glazyrin's upper bound

Define: $\mathrm{dens}(n, \rho) = \sup_{\mathcal{T}} \mathrm{dens}(\mathcal{T}, \rho)$

= Area covered / Total area



SDP bound on Glazyrin's density function

Theorem (Dostert–K.–Oliveira, 2020):

Let $n \geq 3$, $1 < \rho < 3$, and R such that $R > \dfrac{\rho - 1}{2}$.

Let $r : [0,1] \to [(\rho - 1)/2, R]$ be an increasing bijection.

Let also $a : [0,1] \to \mathbb{R}$ be such that $a(u) \geq A_{n,\rho}(r(u))^{1/2}$ for all $u \in [0,1]$, while $a(1) \geq A_{n,\rho}(\infty)^{1/2}$.

SDP bound on Glazyrin's density function

Theorem (Dostert–K.–Oliveira, 2020):

Fix an integer $d > 0$ and for every $k = 0, \ldots, d$;

let $F_k : [0,1]^2 \to \mathbb{R}$ be a kernel.

Write $f(t, u, v) = \sum_{k=0}^{d} F_k(u, v)\, P_k^n(t)$ for $t \in [-1,1]$, $u, v \in [0,1]$,

where $P_k^n(t)$ denotes the Jacobi polynomial of degree k, with parameters $\alpha = \beta = (n-3)/2$, normalised so that $P_k^n(1) = 1$.

SDP bound on Glazyrin's density function

Theorem (Dostert—K.—Oliveira, 2020):

If $f(t, u, v)$ and the kernels $F_k(u, v)$ are such that:

1) every principal submatrix of $F_0 - a \otimes a^*$ is PSD;

2) every principal submatrix of F_k is PSD, $k = 0, \ldots, d$;

3) $f(t, u, v) \leq 0$ whenever $-1 \leq t \leq \dfrac{1 + r(u) + r(v) - r(u)r(v)}{1 + r(u) + r(v) + r(u)r(v)}$;

SDP bound on Glazyrin's density function

Theorem (Dostert—K.—Oliveira, 2020):

Then

$$\mathrm{dens}(n, \rho) \leq \max_{u \in [0,1]} f(1, u, u).$$

SDP bound on Glazyrin's density function

The SDP that we obtain is of similar type to that in the of de Laat, Oliveira, and Vallentin "Upper bounds for packings of spheres of several radii", Forum of Mathematics, Sigma 2 (2014) e23.

We specify the kernels $F_k, k = 0, \ldots, d$, by using either polynomials of degree ≤ 12 or step functions with an adaptive choice of "steps". Then we can use the methods from the above paper. Our solver of choice is SDPA—GMP (Yamashita et al.)

Thank you!

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