Facial Structures of EDM cones and Intersection with Linear Subspaces

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FIELDS

Outline

EDM Cone

let $J = I - \frac{1}{n}ee^T$ be the orthogonal projection on the orthogonal complement of the vector of ones, e.

Definition

We consider $\mathcal{E}^n(r)$ to be the set of EDMs with embedding dimensions not greater than r, that is

$$Y \in \mathcal{E}^n(r) \iff Y \in \mathcal{S}^n_h, -JYJ \succeq 0, \text{ and } \operatorname{rank}(JYJ) \leq r,$$

where $\mathcal{S}^n_h$ is the *hollow subspace* in $\mathcal{S}^n$ defined by $\mathcal{S}^n_h := \{X \in \mathcal{S}^n : \operatorname{diag}(X) = 0\}$.

The EDM of all possible embedding dimension, denoted as $\mathcal{E}^n$ forms a cone, i.e.,

$$\mathcal{E}^n \text{ is a cone in } \mathcal{S}^n.$$

$\mathcal{E}^n(r)$ is called a r-stratum of the EDM cone.

Definition

We let $\mathcal{E}$ be a Euclidean space; and $K \subset \mathcal{E}$ is a convex cone if $K + K \subseteq K, \lambda K \subseteq K, \forall \lambda \geq 0$. A convex cone $F \subseteq K$ is a face of K, $F \trianglelefteq K$, if

$$x, y \in K, \frac{x+y}{2} \in F \implies x, y \in F.$$

The dual cone is $K^* = \{\phi \in \mathcal{E} : \langle \phi, k \rangle \geq 0, \forall k \in K\}$. The *conjugate face,* $F^c = K^* \cap F^\perp$. A face is *exposed* if there exists $\phi \in F^c$ such that $F = \phi^\perp \cap K$. If every face of K is an exposed face, then it is a *facially exposed cone*.

Theorem

Both the PSD cone and the EDM cones are facially exposed.

faces of EDM cone

Theorem (Abdo 2018)

Let $D_1 \in \mathcal{E}^n$. Then

$$face(D_1, \mathcal{E}^n) = \{D \in \mathcal{E}^n : null(-\frac{1}{2}V^T D_1 V) \subseteq null(-\frac{1}{2}V^T DV\}$$

Theorem (Abdo 2018)

Let $D_1 \in \mathcal{E}^n$ and $VV^T = J$ Then

$$face(D_1, \mathcal{E}^n) = \{D \in \mathcal{E}^n : gal(D_1) \subseteq gal(D)\}$$

where $gal(D_1) = null \begin{bmatrix} P_1^T \\ e^T \end{bmatrix}$ *and* $P_1 P_1^T = -\frac{1}{2} J D_1 J$.

Definition

Define the Lindenstrauss mapping/operator that maps a Gram matrix to an EDM matrix

$$\mathcal{K}(G) = \text{diag}(G)e^T + e\,\text{diag}(G)^T - 2G.$$

The generalized inverse is

$$\mathcal{K}^\dagger(D) = -J\,\text{offDiag}(D)J,$$

that maps EMDs to Gram matrices.

Theorem

Let $D_1 \in \mathcal{E}^n(r)$ *and* $V_1\Sigma_r V_1^T = -\frac{1}{2}JD_1J$. *Then*

$$\bar{D} \in face(D_1, \mathcal{E}^n) \iff \bar{D} = \mathcal{K}(V_1\mathcal{S}^r_+ V_1^T) \text{ for some } \mathcal{S}^r_+)$$

A characterization of the r-stratum

Theorem

Let $D_1 \in \mathcal{E}^n(r)$ *and* $\mathrm{range}(U_1) = \mathrm{null}(\begin{bmatrix} V_1^T \\ e^T \end{bmatrix})$ *Then*

$$face(D_1, \mathcal{E}^n) = \{D \in \mathcal{E}^n, \langle D, U_1 U_1^T \rangle = 0\}$$

Similar to the positive semidefinite cone $\mathcal{S}_+^n$ where $\mathcal{S}_+^n(r)$ a union of faces of $\mathcal{S}_+^n$ with rank at most r. For EDM cone, we have

$$\mathcal{K}(V\mathbb{S}_+^r V^T) = \mathcal{E}^n(r), \text{ for any } V \text{ of full column rank .}$$

A conjecture

Conjecture

Given a linear subspace L, if L passes through the relative interior of $\mathcal{E}^n$ and the interior of $\mathcal{E}^n(r)$. Then L intersects $\mathcal{E}^n(r)$ properly, i.e.,

$$\dim(\mathcal{E}^n(r) \cap L) = \dim(\mathcal{E}^n(r)) - \operatorname{codim}(L).$$

It could have lower dimension if

1. (a) L does not intersect the interior of $\mathcal{E}^n(r)$
2. (b) It intersects the interior of $\mathcal{E}^n(r)$ but it is tangent to $\mathcal{E}^n(r)$

If it is tangent to $\mathcal{E}^n(r)$, then it is tangent to $\mathcal{E}^n(r+1)$?

Cycle packing space

Definition (Packing Space and Projection)

Given a graph $G = (V, E)$ with $|E| \geq |V|$, and the graph is connected, consider the linear system in $|V| + \binom{|V|}{2}$ variables x_i and y_{ij}, for $i \in V$ and $i \neq j$, given by the equations $x_i + x_j = y_{ij}$ for $ij \in E$, the solutions to this system span a $\binom{|V|}{2} - (|E| - |V|)$ dimensional space.
Then the projection of this *packing space* S using $(x, y) \to y$ (eliminate the x_i's) is denoted as S' of whose dimension is at most $\binom{|V|}{2} - (|E| - |V|)$.

Lemma

Assuming the graph is connected and $|E| \geq |V|$. The projection of the packing space has dimension $\binom{|V|}{2} - (|E| - (|V| - a_G))$, i.e., the co-dimension $|E| - (|V| - a_G)$, where a_G is the (column) rank deficit of A_x, where the linear system is $A = [A_x, A_y]$.

Consider a 4-cycle, with vertices being v_1, v_2, v_3, v_4 and edges $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}$, the linear system is given by the following

$$A = [A_x,\ A_y] = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

where A_x is the 4×4 coefficient matrix for $[x_1, x_2, x_3, x_4]$ and A_y is the coefficient matrix for $[y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}]$. The solutions to this system span a $\binom{|V|}{2} - (|E| - |V|) = 6 - (4 - 4) = 6$ dimensional space.
However since rank $A_x = 3$ and $y_{14} = y_{12} - y_{23} + y_{34}$, the projection $(x, y) \rightarrow y$ has dimension 5 which is less than $\binom{|V|}{2} - (|E| - |V|)$ even G has 4 edges and 4 vertices. The co-dimension is $6 - 5 = 1$.

Laman graph and cycle packing space

Theorem (MAIN0)

All planar minimally rigid (Laman) graphs have rank deficit 0.

Theorem (MAIN)

Let E be the edge set of a minimal rigid planar graph (Laman) G and $|V| = n$. Therefore $|E| \geq V$. Then the dimension $\mathcal{E}^n(2) \cap S'$ of dimension is $2|V| - 3 - (|E| - |V|)$, i.e.,

$$\dim(\mathcal{E}^n(2) \cap S') = \dim(\mathcal{E}^n(2)) - \text{codim}(S') = |V|$$

Since G is a Laman graph, we have $|E| = 2|V| - 3$, hence $2|V| - 3 - (|E| - |V|) = |V|$

proof of Theorem Main0

Proof.

1. Using induction, the induction hypothesis (IH) is the following: assume the base case is a triangle, and A_x has no rank deficit,. The dependent edges are marked. There is exactly 1 cycle C of unmarked edges and it is odd.
2. We use a planar minimally rigid graph inductive construction (by Fekete et al) using only plane vertex splits. In each step of this construction, an edge uv in a planar graph is split into two edges uv_1 and uv_2, adding the edge v_1v_2 and splitting the neighbors $w_1, \ldots, w_i, w_{i+1}, \ldots, w_k$ of v by making $w_1, \ldots, w_i$ neighbors of v_1 and the rest neighbors of v_2 while maintaining planarity. It is known that every planar minimally rigid graphs can be generated from an edge by a sequence of vertex splits [Fekete].
3. In this case, 1 vertex, say v_2, and and 2 edges, say v_1v_2 and uv_2 are added at each stage.

□

Proof Continue...

Claim is that $v_1 v_2$ can be marked dependent, and the odd cycle C of unmarked edges can be either retained or modified into another odd cycle, no other new cycles of unmarked edges are added, and hence IH holds.
There are 3 cases:

1. If C did not previously contain v, then (after marking $v_1 v_2$ dependent,) C is retained, no new cycles, so induction hypothesis still holds.
2. If the odd cycle C contains v but not u,
   substitute v in C by the path $v_1 u v_2$ and the modified C is still odd, no new cycles, so IH still holds.
3. If C contains both u and v, then the odd cycle C remains and no new cycles are formed, hence IH still holds.

Proof of Theorem Main1

1. Recall in the proof of Theorem Main0, at each step of the vertex splitting construction, the edge $v_1 v2$ is marked as dependent, which means the codimension of S' is increased by 1, however at the same time, the edge $v_1 v_2$ is independent in the rigidity matroid of G, i.e., the length of $v_1 v_2$ can change freely in a local neighborhood without changing the length of other edges, which means $v_1 v_2$ adds one more dimension to $\mathcal{E}^n(2)$.
2. Hence after each vertex splitting, let G' to be obtained from G by vetex splitting. Also let $V' = |V| + 1$ and $|E'| = |E| + 2$ by the vertex splitting operation, then

$$\dim(\mathcal{E}^{n+1}(2) \cap S'(G')) = 2|V'| - 3 - 1 - (|E| - |V|) = 2|V'| - 3 - (|E'| - |V'|)$$

It is also east see that the dimension $M_2 \cap S'$ of dimension is $2|V| - 3 - (|E| - |V|)$ at the base of the induction hypothesise (A trianle), therefore the theorem is proved.

Relation with the generic cycle packing problem

Definition

A planar graph is said to be triangulated (also called maximal planar) if the addition of any edge to results in a nonplanar graph.

Theorem (Koebe-Andreev-Thurston)

For any planar graph G, there is a circle packing representation, after completing the planar graph into a triangulation. The graph G is called the tangency graph of the circle packing.

Genericity of circle packing realization

Question

Given a set of generic radii $r_i, i = 1..., |V|$ (coins), what are the possible tangency graphs?

Conjecture

[Connelly-Gortler-Theran] Planar Laman graphs(which after triangulation have the KAT (Koebe Andreev Thurston) pack-ing realization) do have a generic radius 2-dimension packing realization

Theorem (Connelly-Gortler-Theran)

If a packing has generic radii, then the allowed motions are all rigid body motions if and only if the packing has exactly $2n - 3$ contacts.

1. We have proved this conjecture without non-edge packing costraints.
2. The proof is valid for all 2D minimally rigid graphs that can be obtained using vertex splitting and Henneberg 1 extension, which includes planar minimally rigid (Laman) graphs, but maybe more.

Definition

Consider adding on the inequalities $x_i + x_j \leq y_{ij}$ for $ij \notin E$ to S', the resulting convex cone is denoted as and T'.

1. If the following holds

$$\dim(\mathcal{E}^n(2) \cap S') = \dim(\mathcal{E}^n(2) \cap T') = |V|.$$

Then it will imply the conjecture is true.

Thank You for Listening!

References I

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