Classical and Multivariable Toeplitz Matrices: Completions and Other Aspects

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Based on joint works with co-authors Jeffrey Geronimo, Selcuk Koyuncu, Stefan Sremac, Henry Wolkowicz and Chung Wong.

SIMONS FOUNDATION

Stable polynomials and Toeplitz operators

Main ingredients for this talk:

- **Stable polynomials**

$$p(z_1, \ldots, z_d) = \sum_{0 \le k_j \le n_j} p_{k_1,\ldots k_d} z_1^{k_1} \cdots z_d^{k_d},$$

  with $p(z_1, \ldots, z_d) \neq 0$ when $|z_1| \le 1, \ldots, |z_d| \le 1$.

- **(Multivariable) Toeplitz operators**

$$T_\Lambda = (c_{(k_1,\cdots,k_d)-(\ell_1,\cdots,\ell_d)})_{(k_1,\cdots,k_d),(\ell_1,\cdots,\ell_d)\in\Lambda}.$$

  For example, if $\Lambda = \{0,1\} \times \{0,1,2\}$,

$$T_\Lambda = \left(\begin{array}{ccc|ccc} c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\ c_{01} & c_{00} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} \\ c_{02} & c_{01} & c_{00} & c_{-1,2} & c_{-1,1} & c_{-1,0} \\ \hline c_{10} & c_{1,-1} & c_{1,-2} & c_{00} & c_{0,-1} & c_{0,-2} \\ c_{11} & c_{10} & c_{1,-1} & c_{01} & c_{00} & c_{0,-1} \\ c_{12} & c_{11} & c_{10} & c_{02} & c_{01} & c_{00} \end{array}\right).$$

Overview

Part 1: Gohberg-Semencul formula and a two-variable generalization.

- Selcuk Koyuncu and Hugo J. Woerdeman, The inverse of positive definite two-level Toeplitz operator matrices, *Operator Theory: Adv. Appl.* 218 (2012), 387–401.

Part 2: When are maximum determinant completions Toeplitz?

- S. Sremac, H. Wolkowicz, and H. J. Woerdeman, Maximum determinant positive definite Toeplitz completions. *OT: Adv. Appl.* 271 (2018), 421–441.

Part 3: The autoregressive filter problem in two and more variables

- J. S. Geronimo and H. J. Woerdeman, Positive Extensions, Fejér-Riesz Factorization and Autoregressive Filters in Two Variables, *Ann. Math.* 160 (2004), no. 3, 839–906.
- J. S. Geronimo, H. J. Woerdeman, C. Y. Wong, *The autoregressive filter problem for multivariable degree one symmetric polynomials*, arXiv:2101.00525 (2021).

Completion problem to look out for ...

Find the unknown red entries

$$\begin{pmatrix} c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\ c_{01} & c_{00} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} \\ c_{02} & c_{01} & c_{00} & c_{-1,2} & c_{-1,1} & c_{-1,0} \\ c_{10} & c_{1,-1} & c_{1,-2} & c_{00} & c_{0,-1} & c_{0,-2} \\ c_{11} & c_{10} & c_{1,-1} & c_{01} & c_{00} & c_{0,-1} \\ c_{12} & c_{11} & c_{10} & c_{02} & c_{01} & c_{00} \end{pmatrix}$$

resulting in a **positive definite matrix** with

$$\operatorname{rank} \begin{pmatrix} c_{1,-1} & c_{1,-2} & c_{0,-1} & c_{0,-2} \\ c_{10} & c_{1,-1} & c_{00} & c_{0,-1} \\ c_{11} & c_{10} & c_{01} & c_{00} \end{pmatrix} \leq 2 \ \ (\text{or } = 2).$$

Stable polynomials and Toeplitz operators

Main ingredients for this talk:

- **Stable polynomials**

$$p(z) = p(z_1, \ldots, z_d) = \sum_{0 \le k_j \le n_j} p_{k_1,\ldots k_d} z_1^{k_1} \cdots z_d^{k_d} = \sum_{k \in \underline{n}} p_k z^k,$$

  with $p(z_1, \ldots, z_d) \neq 0$ when $z \in \overline{\mathbb{D}}^d$, the closed polydisk.

- **(Multivariable) Toeplitz operators**

$$T_\Lambda = (c_{(k_1,\cdots,k_d)-(\ell_1,\cdots,\ell_d)})_{(k_1,\cdots,k_d),(\ell_1,\cdots,\ell_d)\in\Lambda}.$$

  For example, if $\Lambda = \{0,1\} \times \{0,1,2\}$,

$$T_\Lambda = \left(\begin{array}{ccc|ccc}
c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\
c_{01} & c_{00} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} \\
c_{02} & c_{01} & c_{00} & c_{-1,2} & c_{-1,1} & c_{-1,0} \\
\hline
c_{10} & c_{1,-1} & c_{1,-2} & c_{00} & c_{0,-1} & c_{0,-2} \\
c_{11} & c_{10} & c_{1,-1} & c_{01} & c_{00} & c_{0,-1} \\
c_{12} & c_{11} & c_{10} & c_{02} & c_{01} & c_{00}
\end{array}\right).$$

Part 1

One variable stable polynomials

Simple observation: $p(z) = p_0 + p_1 z$ is stable (= has no roots in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$) if and only if $|p_0| > |p_1|$.

Theorem (Schur-Cohn criterion)

$p(z) = p_0 + p_1 z + \cdots + p_n z^n$ is stable iff

$$S_n = \begin{bmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{bmatrix}$$

$$- \begin{bmatrix} \overline{p_n} & & & \\ \overline{p_{n-1}} & \overline{p_n} & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & \overline{p_n} \end{bmatrix} \begin{bmatrix} p_n & p_{n-1} & \cdots & p_1 \\ & p_n & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & p_n \end{bmatrix}$$

is positive definite. **Comment:** *With $p_n = 0$, S_n^{-1} is Toeplitz!*

Trench '64, Gohberg-Semencul '72

Theorem

If $T = (c_{i-j})_{i,j=0}^{n-1} > 0$ *then*

$$T^{-1} = \begin{bmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & p_{n-1} & \cdots & p_1 \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 0 \end{bmatrix}.$$

Also, $p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1}$ has no roots in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ [Szegö, 1920]

Relationship: $c_k = \widehat{\frac{1}{|p(z)|^2}}(k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ikt}}{|p(e^{it})|^2} dt =$ Fourier coeff.

Sketch of proof based on Schur complement

Lemma. $\begin{pmatrix} (A - BC^{-1}B^*)^{-1} & * \\ * & * \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. $\square$

Write $\frac{1}{p(z)} = q_0 + q_1 z + q_2 z^2 + \cdots, |z| \leq 1$. Then, since $c_k = \widehat{\frac{1}{|p(z)|^2}}(k)$, we have

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \ddots \\ c_2 & c_1 & c_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} \overline{q_0} & \overline{q_1} & \overline{q_2} & \cdots \\ & \overline{q_0} & \overline{q_1} & \ddots \\ & & \overline{q_0} & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} q_0 & & & \\ q_1 & q_0 & & \\ q_2 & q_1 & q_0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \ddots \\ c_2 & c_1 & c_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ p_2 & p_1 & p_0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \overline{p_0} & \overline{p_1} & \overline{p_2} & \cdots \\ & \overline{p_0} & \overline{p_1} & \ddots \\ & & \overline{p_0} & \ddots \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Now $A = \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{pmatrix} \begin{pmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{pmatrix},$

$$B = \begin{pmatrix} p_0 & & & & & \\ p_1 & p_0 & & & 0 & \cdots \\ \vdots & \vdots & \ddots & & & \\ p_{n-1} & p_{n-2} & \cdots & p_0 & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \overline{p_{n-2}} & \overline{p_{n-1}} & 0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} 0 & & & & \\ \overline{p_{n-1}} & 0 & & 0 & \cdots \\ \vdots & \ddots & \ddots & & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 & \end{pmatrix} \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \text{ and}$$

$$BC^{-1}B^* = \begin{pmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 \end{pmatrix} \begin{pmatrix} 0 & p_{n-1} & \cdots & p_1 \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 0 \end{pmatrix}.$$

Inverses of two-variable Toeplitz matrices

For example

$$\left(\begin{array}{ccc|cc} c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} \\ c_{01} & c_{00} & c_{0,-1} & c_{-1,1} & c_{-1,0} \\ c_{02} & c_{01} & c_{00} & c_{-1,2} & c_{-1,1} \\ \hline c_{10} & c_{1,-1} & c_{1,-2} & c_{00} & c_{0,-1} \\ c_{11} & c_{10} & c_{1,-1} & c_{01} & c_{00} \end{array}\right)^{-1} = ??$$

Theorem ([Koyuncu & W., Operator Theory: Adv. Appl., 2012])

Let $p(z) = \sum_{k \in \underline{n}} p_k z^k$ be a two-variable stable polynomial and let c_k denote the Fourier coefficients of $\frac{1}{|p|^2}$. Then

$$[(c_{m-l})_{m,l \in \underline{n} \setminus \{n\}}]^{-1} = AA^* - B^*B - C_1^* D_1^{-1} C_1 - C_2^* D_2^{-1} C_2,$$

where $A = (p_{k-l})_{k,l \in \underline{n}}$, $\quad B = (p_{k-l})_{k \in n + \underline{n},\, l \in \underline{n}}$, and C_1, C_2, D_1, D_2 are also given in terms of p_k, $k \in \underline{n}$. Here D_1 and D_2 are banded one variable Toeplitz operators.

Part 2

Toeplitz matrix completions

Given *Toeplitz real symmetric partial matrix*, e.g.

$$\mathcal{T} = \begin{bmatrix} 8 & 4 & ? & 1 \\ 4 & 8 & 4 & ? \\ ? & 4 & 8 & 4 \\ 1 & ? & 4 & 8 \end{bmatrix}.$$

We are interested in *positive definite completions*, e.g.

$$T = \begin{bmatrix} 8 & 4 & 0 & 1 \\ 4 & 8 & 4 & 1 \\ 0 & 4 & 8 & 4 \\ 1 & 1 & 4 & 8 \end{bmatrix}, T^\star = \begin{bmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 1 & 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{bmatrix}^{-1}.$$

Among all positive definite completions there is a unique one with *maximum determinant*; it is the unique one with zeroes in the inverse in positions corresponding to unknowns [Grone, Johnson, Sa, Wolkowicz, 1984].

Question: For what patterns is this unique completion Toeplitz?

Previous results

A *pattern* $P \subseteq \{1, \ldots, n-1\}$ indicates the prescribed diagonals in the lower triangular part. E.g., $n = 4$,

$$P = \{1, 3\} \Leftrightarrow \begin{bmatrix} * & * & ? & * \\ * & * & * & ? \\ ? & * & * & * \\ * & ? & * & * \end{bmatrix}, \quad P = \{1\} \Leftrightarrow \begin{bmatrix} * & * & ? & ? \\ * & * & * & ? \\ ? & * & * & * \\ ? & ? & * & * \end{bmatrix}.$$

- [Dym & Gohberg, 1981] Banded Toeplitz partial positive definite matrices have a Toeplitz maximum determinant completion. Here $P = \{1, 2, \ldots, r\}$.
- [Naevdal, 1997] If the unknown diagonal is one but last one and a positive semidefinite completion exists, then a Toeplitz one as well. Here $P = \{1, 2, \ldots, n-3, n-1\}$.
- [He & Ng, 2005] **Conjecture**: in the cycle case if a positive semidefinite completion exists, then a Toeplitz one as well. Here $P = \{k, n-k\}$.

Observation

If $P = kP'$, then the partial Toeplitz matrix with pattern P is permutation similar to a k block diagonal of (almost) the same partial Toeplitz matrices with pattern P'.

E.g., $n = 7, P = \{2, 4\}$, then

$$\begin{bmatrix} a & ? & b & ? & c & ? & ? \\ ? & a & ? & b & ? & c & ? \\ b & ? & a & ? & b & ? & c \\ ? & b & ? & a & ? & b & ? \\ c & ? & b & ? & a & ? & b \\ ? & c & ? & b & ? & a & ? \\ ? & ? & c & ? & b & ? & a \end{bmatrix} \sim \begin{bmatrix} a & b & c & ? & ? & ? & ? \\ b & a & b & c & ? & ? & ? \\ c & b & a & b & ? & ? & ? \\ ? & c & b & a & ? & ? & ? \\ ? & ? & ? & ? & a & b & c \\ ? & ? & ? & ? & b & a & b \\ ? & ? & ? & ? & c & b & a \end{bmatrix}.$$

For the maximum determinant the unknowns outside the block diagonal are all 0.

Main characterization of patterns

For a positive definite completable partial Toeplitz matrix $\mathcal{T}$, we denote the unique maximum determinant positive definite completion by $T^\star$.

Theorem ([Sremac, Wolkowicz, W., Operator Theory: Adv. Appl., 2018])

Let $\emptyset \neq P \subseteq \{1, \ldots, n-1\}$ denote a pattern. The following are equivalent.

1. *For every positive definite completable partial Toeplitz matrix $\mathcal{T}$ with pattern P, the matrix $T^\star$ is Toeplitz.*
2. *There exist $r, k \in \mathbb{N}$ such that P has one of the three forms:*
   - $P_1 := \{k, 2k, \ldots, rk\}$,
   - $P_2 := \{k, 2k, \ldots, (r-2)k, rk\}$, *where* $n = (r+1)k$,
   - $P_3 := \{k, n-k\}$.

Crucial characteristic of patterns

$$T^{\star-1} = \begin{bmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & p_{n-1} & \cdots & p_1 \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 0 \end{bmatrix}.$$

can have the correct zero structure $\Leftrightarrow$

P is one of the three types of patterns.

Consequence

Theorem ([Sremac, Wolkowicz, W., Operator Theory: Adv. Appl., 2018])

Consider the patterns

- $P_1 := \{k, 2k, \dots, rk\}$,
- $P_2' := \{k, 2k, \dots, (r-2)k, rk\}$,
- $P_3' := \{k, r\}$ *where* $n \geq k + r$.

If $\mathcal{T}$ *is an* $n \times n$ *positive semidefinite completable partial Toeplitz matrix with a pattern in the set* $\{P_1, P_2', P_3'\}$, *then* $\mathcal{T}$ *has a Toeplitz positive semidefinite completion.*

Idea of proof. Adjusting the size n reduces it to the previous theorem. Applying the previous theorem leads now to a banded pattern. Now use [Dym & Gohberg, 1981].

Note. This proves the conjecture of [He & Ng, 2005].

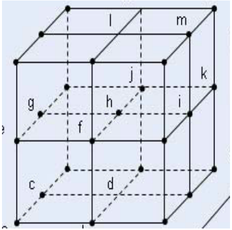
Question. Are these all patterns where a positive semidefinite completion implies the existence of a Toeplitz one?

Part 3

Identification for stationary stochastic processes

Consider stochastic process $X_m = X_{(m_1,\dots,m_d)}$ depending on d discrete variables defined on a fixed probability space. We shall assume that the random variables X_m are *centered*, i.e., their means $E(X_m)$ equal zero. The process $X = (X_m)_{m\in\mathbb{Z}^d}$ is called *stationary* on $\mathbb{Z}^d$ if for $m, k \in \mathbb{Z}^d$ we have that

$$E(X_m^* X_k) = E(X_{m+p}^* X_{k+p}) =: c_{m-k} \text{ for all } p \in \mathbb{Z}^d.$$



At each node a stochastic variable.

Stationary = correlation only depends on position relative to one another.

For $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ we let $\underline{n} = \prod_{j=1}^d \{0, \dots, n_j\}$.
The process X is called *auto-regressive* if there exist complex numbers $a_k, k \in \underline{n} \setminus \{0\}$, such that for every t,

$$X_t + \sum_{k \in \underline{n}, k \neq 0} a_k X_{t-k} = E_t, \qquad t \in \mathbb{Z}^d, \tag{1}$$

where $\{E_k\ ; k \in \mathbb{Z}^d\}$ is a white noise zero mean process with variance σ^2. Let H be the standard half-space in $\mathbb{Z}^d$; that is

$$H = \{(k_1, \dots, k_d) \in \mathbb{Z}^d : \exists\, j : \ k_1 = \cdots = k_{j-1} = 0 \ \&\ k_j > 0\}.$$

The process is said to be *causal* if there is a solution to (1) of the form

$$X_t = \sum_{k \in H \cup \{0\}} \phi_k E_{t-k}, \ \ t \in \mathbb{Z}^d, \tag{2}$$

with $\sum_{k \in H \cup \{0\}} |\phi_k| < \infty$; this happens if and only if the polynomial

$$\tilde{p}(z) = 1 + \sum_{k \in \underline{n}, k \neq 0} \overline{a}_k z^k$$

has no roots in the closed d-disk; i.e., $\tilde{p}(z)$ is *stable polynomial*.

Multivariate autoregressive filter design problem

The *multivariate autoregressive filter design problem* is the following.

- Given are covariances

$$c_k = E(X_0^* X_k), \qquad k \in \underline{n}.$$

  What conditions must the covariances satisfy in order that these are the covariances of a causal autoregressive process?
- In that case, how does one compute the filter coefficients a_k, $k \in \underline{n} \setminus \{0\}$ and σ^2?

In other words:

- Given $c_k = \overline{c_{-k}} \in \mathbb{C}$, $k \in \underline{n}$.
- Find $p(z) = \sum_{k \in \underline{n}} p_k z^k$ so that $p \neq 0$ on $\overline{\mathbb{D}}^d$ and $\widehat{\frac{1}{|p|^2}}(k) = c_k$, $k \in \underline{n}$.

Classical result (Toeplitz, Carathéodory, 1910s)

Given $c_k = \overline{c_{-k}} \in \mathbb{C}$, $k \in \{0, \ldots, n\}$. They are the Fourier coefficients of a spectral density function as above if and only if

$$T_n = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{pmatrix} > 0.$$

Yule-Walker, 1950s: Find the polynomial $p(z)$ via the equation

$$\overline{p_0} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = T_n^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In general, we have

$$[(c_{k-l})_{k,l\in\underline{n}}]^{-1}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}=\overline{p_0}\,\mathrm{col}(p_k)_{k\in\underline{n}},$$

where $\underline{n}=\prod_{j=1}^{d}\{0,\ldots,n_j\}$.

- Need coefficients $c_m, m\in\underline{n}-\underline{n}=\prod_{i=1}^{d}\{-n_i,\ldots,n_i\}$.
- Due to symmetry $c_k=\overline{c_{-k}}$, this results in about $\frac{1}{2}\prod_{i=1}^{d}(2n_i)=2^{d-1}\prod n_i$ parameters.
- The polynomial has $\approx\prod n_i$ parameters p_k, $k\in\underline{n}$.
- So about 2^{d-1} as many parameters on the left compared to the right. Only when $d=1$ it matches!

Problem: Identify the relationship between the c_k's, $k\in\underline{n}-\underline{n}$
Or: Given c_k, $k\in\underline{n}$. Find remaining c_k, $k\in(\underline{n}-\underline{n})\setminus(\underline{n}\cup-\underline{n})$.

In two variables

Given $c_k = \overline{c_{-k}} \in \mathbb{C}$, $k = (k_1, k_2) \in \{0, \ldots, n_1\} \times \{0, \ldots n_2\} =: \underline{n}$

Solution involves $T_{\underline{n}} := (c_{k-l})_{k,l \in \underline{n}}$; $\underline{n} = \{0 \ldots n_1\} \times \{0 \ldots n_2\}$, which contains missing entries (in red below).

Example: $d = 2$, $\underline{n} = \{0, 1\} \times \{0, 1, 2\}$ with lexicographical order:

$$T_{\underline{n}} = \begin{pmatrix} c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\ c_{01} & c_{00} & c_{0,-1} & {\color{red}c_{-1,1}} & c_{-1,0} & c_{-1,-1} \\ c_{02} & c_{01} & c_{00} & {\color{red}c_{-1,2}} & {\color{red}c_{-1,1}} & c_{-1,0} \\ c_{10} & {\color{red}c_{1,-1}} & {\color{red}c_{1,-2}} & c_{00} & c_{0,-1} & c_{0,-2} \\ c_{11} & c_{10} & {\color{red}c_{1,-1}} & c_{01} & c_{00} & c_{0,-1} \\ c_{12} & c_{11} & c_{10} & c_{02} & c_{01} & c_{00} \end{pmatrix}.$$

Solution in two variables

Theorem ([Geronimo & W., Ann. of Math., 2004])

The given data $c_k = \overline{c_{-k}}$, $k \in \{0, \dots, n_1\} \times \{0, \dots, n_2\} =: \underline{n}$ *are the Fourier coefficients of a spectral density function as above iff one can find* c_k, $k \in \{1, \dots, n_1\} \times \{-n_2, \dots, -1\}$ *so that*

- $T_{\underline{n}} > 0$
- *rank* $(c_{k-\ell})_{\substack{k\in\{1,\dots,n_1\}\times\{0,\dots,n_2\} \\ \ell\in\{0,\dots,n_1\}\times\{1,\dots,n_2\}}} = n_1 n_2$
- *or, equivalently,* $[(c_{k-\ell})_{k,\ell\in\underline{n}\setminus\{0\}}]^{-1}_{\substack{\{1,\dots,n_1\}\times\{0\} \\ \{0\}\times\{1,\dots,n_2\}}} = 0.$

An example: Take $c_{00} = 1, c_{01} = c_{10} = c_{11} = 0.9.$

$$\begin{pmatrix} 1 & 0.9 & 0.9 & 0.9 \\ 0.9 & 1 & \overline{x} & 0.9 \\ 0.9 & x & 1 & 0.9 \\ 0.9 & 0.9 & 0.9 & 1 \end{pmatrix}$$

$x = c_{1,-1} = 0.81 \Rightarrow$ positive definite $\Rightarrow$ solution exists. $\square$

Important idea in the proof

Write $p(z_1, z_2) = p_0(z_1) + \cdots + p_{n_2}(z_1) z_2^{n_2}$ and use the special zero structure of the Fejér-Riesz factorization of

$$T^{-1}(z_1) = \begin{bmatrix} p_0(z_1) & & \\ \vdots & \ddots & \\ p_{n_2}(z_1) & \cdots & p_0(z_1) \end{bmatrix} \begin{bmatrix} \overline{p_0}(1/z_1) & \cdots & \overline{p_{n_2}}(1/z_1) \\ & \ddots & \vdots \\ & & \overline{p_0}(1/z_1) \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & \\ \overline{p_{n_2}}(1/z_1) & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1}(1/z_1) & \cdots & \overline{p_{n_2}}(1/z_1) & 0 \end{bmatrix} \begin{bmatrix} 0 & p_{n_2}(z_1) & \cdots & p_1(z_1) \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n_2}(z_1) \\ & & & 0 \end{bmatrix}$$

$$= \begin{pmatrix} p_0(z_1) & 0 \\ \tilde{p}(z_1) & M(z_1) \end{pmatrix} \begin{pmatrix} \overline{p_0}(1/z_1) & \tilde{p}^*(1/z_1) \\ 0 & M^*(1/z_1) \end{pmatrix}.$$

Positive definite rank structured completion problem

Find the unknown red entries

$$T = \begin{pmatrix} c_{00} & c_{0,-1} & c_{0,-2} & c_{-1,0} & c_{-1,-1} & c_{-1,-2} \\ c_{01} & c_{00} & c_{0,-1} & c_{-1,1} & c_{-1,0} & c_{-1,-1} \\ c_{02} & c_{01} & c_{00} & c_{-1,2} & c_{-1,1} & c_{-1,0} \\ c_{10} & c_{1,-1} & c_{1,-2} & c_{00} & c_{0,-1} & c_{0,-2} \\ c_{11} & c_{10} & c_{1,-1} & c_{01} & c_{00} & c_{0,-1} \\ c_{12} & c_{11} & c_{10} & c_{02} & c_{01} & c_{00} \end{pmatrix} = \begin{pmatrix} c_{00} & * \\ * & \widetilde{T} \end{pmatrix}$$

resulting in a **positive definite matrix** with

$$\operatorname{rank} \begin{pmatrix} c_{1,-1} & c_{1,-2} & c_{0,-1} & c_{0,-2} \\ c_{10} & c_{1,-1} & c_{00} & c_{0,-1} \\ c_{11} & c_{10} & c_{01} & c_{00} \end{pmatrix} \leq 2 \ \ (\text{or } = 2),$$

or, equivalently,

$$\widetilde{T}^{-1} = \begin{pmatrix} * & * & 0 & * & * \\ * & * & 0 & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}.$$

Three or more variables.

Experimental results: no obvious generalizations work.

Idea: Try the simplest case $p(z) = p_0 + p_1(z_1 + \cdots + z_d)$.

We only need to determine 2 parameters, p_0 and p_1! Let $\Lambda = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$. Then

$$(c_{m-l})_{m,l\in\Lambda} = \begin{bmatrix} a & \bar{b} & \bar{b} & \cdots & \bar{b} \\ b & a & c & \cdots & c \\ b & c & a & \cdots & c \\ \vdots & \vdots & & \ddots & \vdots \\ b & c & c & \cdots & a \end{bmatrix},$$

where

$$a = \widehat{\frac{1}{|p|^2}}(0,0,\ldots,0), b = \widehat{\frac{1}{|p|^2}}(1,0,\ldots,0), c = \widehat{\frac{1}{|p|^2}}(1,-1,0,\ldots,0).$$

Question: What is the relation between a, b, and c?

Theorem ([Geronimo, Wong & W., 2021])

A stable polynomial $p(z) = p_0 + p_1(z_1 + \cdots + z_d)$ *exists with* $a = c_{0\cdots 0}$, $b = c_{10\cdots 0}$, $c = c_{1,-1,0\cdots 0}$ *iff positive definiteness and*

$$\frac{a(a+(d-1)c)}{a^2+(d-1)ac+d|b|^2} = \int_{[0,2\pi]^{d-2}} \frac{1}{(2\pi)^{d-2} g(t_3,\ldots,t_d)^{\frac{1}{2}}} dt_3 \cdots dt_d,$$

where $g(t_3,\ldots,t_d)$ *equals*

$$\left(1 - \frac{2|b|\sum_{3\le j\le d}\cos t_j}{a+(d-1)c} + \frac{|b|^2\sum_{3\le j,k\le d}\cos(t_j-t_k)}{(a+(d-1)c)^2}\right) \times$$

$$\left(1 - \frac{2|b|\sum_{3\le j\le d}\cos t_j}{a+(d-1)c} + \frac{|b|^2(-4+\sum_{3\le j,k\le d}\cos(t_j-t_k))}{(a+(d-1)c)^2}\right).$$

Need $|b| < (1-\frac{1}{\gamma_d})a$, where $\gamma_d = \sum_{n=0}^{\infty}\sum_{\sum n_i = n}\binom{n}{n_1,\ldots,n_d}^2 d^{-2n}$.

Comparing the cases $d = 1, 2, 3, 4$

- $d = 1$: $|b| < a$.
- $d = 2$: $|b| < a$ and $c = \frac{|b|^2}{a}$ [Geronimo & W., 2004].
- $d = 3$: $|b| < a$ and solve c from
  $\frac{a(a+2c)}{a^2+2ac-3|b|^2} = \frac{(a+2c)^2}{(a+2c)^2-3|b|^2}\ {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}; \frac{27|b|^4((a+2c)^2-|b|^2)}{((a+2c)^2-3|b|^2)^3}\right)$,
  where ${}_2F_1$ is the Gauss hypergeometric function
  $$ {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$
  and $(q)_n$ denotes the rising factorial $q(q+1)\cdots(q+n-1)$.
- $d = 4$: $|b| < 0.442238677a$ and solve c from

$\frac{a(a+3c)}{a^2+3ac+4|b|^2} = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{1}{|1-s(z_3+z_4)|} \frac{1}{\sqrt{|1-s(z_3+z_4)|^2-4|s|^2}} \frac{dz_3}{z_3} \frac{dz_4}{z_4}$,
where $s = \frac{b}{a^2+3ac-4|b|^2}$.

Remark: The expressions for $d \geq 3$ are transcendental over $\mathbb{Q}(x)$.

THANK YOU!

References

Part 1: Gohberg-Semencul formula and a two-variable generalization.

- Selcuk Koyuncu and Hugo J. Woerdeman, The inverse of positive definite two-level Toeplitz operator matrices, *Operator Theory: Adv. Appl.* 218 (2012), 387–401.

Part 2: When are maximum determinant completions Toeplitz?

- S. Sremac, H. Wolkowicz, and H. J. Woerdeman, Maximum determinant positive definite Toeplitz completions. *OT: Adv. Appl.* 271 (2018), 421–441.

Part 3: The autoregressive filter problem in two and more variables

- J. S. Geronimo and H. J. Woerdeman, Positive Extensions, Fejér-Riesz Factorization and Autoregressive Filters in Two Variables, *Ann. Math.* 160 (2004), no. 3, 839–906.
- J. S. Geronimo, H. J. Woerdeman, C. Y. Wong, *The autoregressive filter problem for multivariable degree one symmetric polynomials*, arXiv:2101.00525 (2021).