

On extensions classes of an object
is a filtered Tannakian category

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(Joint work with K. Murty)

§1. Introduction :

Suppose M = a mixed motive over
a number field $K \subset \mathbb{C}$

M Tannakian $G^{\text{mot}}(M)$
 \rightsquigarrow
 formalism motivic Galois
 group of M

Grothendieck's period conjecture :

$\dim G^{\text{mot}}(M) =$ tr. deg. of the field
(conjecturally) generated/ \mathbb{Q} by periods
of M

W. wt filtration on motives

$\text{Gr}^W M$ = Direct sum of pure motives
a semisimple object

There is an exact sequence

$$1 \rightarrow U^{\text{not}}(M) \rightarrow G^{\text{not}}(M) \rightarrow G^{\text{not}}(\text{Gr}^W M) \rightarrow 1$$

unpotent radical reductive
of $G^{\text{not}}(M)$

- $U^{\text{not}}(M)$ is related to extensions
in the category generated by M .
- A recent result of Deligne (2014, in a paper
by Jossen) on the Mumford-Tate conjecture for
(1 -motives) gives a description of $U^{\text{not}}(M)$
in terms of the extensions

$$0 \rightarrow W_p M \rightarrow M \rightarrow M/W_p M \rightarrow 1.$$

Plan :

§ 2 Background on Tannakian categories

§ 3 Deligne's result

§ 4 - § 6 On the joint work with Kumar

§ 2. Background

§ 2.1 Brief reminder on Tannakian formalism

Recall : A (neutral) Tannakian cat. / a field K of char. 0 is a category T which enjoys the nice properties of the category of f.d. representations of a group.

More precisely,

- T is abelian, K -linear

- T has a \otimes structure

- There are internal Hom's:

$$\forall M, N \in \text{ob}(T) \exists \underline{\text{Hom}}(M, N) \in \text{ob}(T)$$

- Setting $M^\vee := \underline{\text{Hom}}(M, 1)$, $M^{\vee\vee} \cong M$

- $\text{End}(1) = K$

Moreover, there is a fiber functor

(i.e. an exact, faithful, K -linear, tensor functor)

$$T \rightarrow \text{Mod}_K.$$

Fund. Theorem: Suppose T is Tannakian,

as a fiber functor. Then

$$\underline{\text{Aut}}^{\otimes}(\omega) : K\text{-alg} \rightarrow \text{Groups}$$

$R \mapsto$ automorphisms of

$$T \xrightarrow{\omega} \text{Mod}_K \rightarrow \text{Mod}_R$$

respecting \otimes structures

is an affine group scheme / K ,
and

$$T \rightarrow \text{Rep}(\underline{\text{Aut}}^{\otimes}(\omega)) \quad (\text{cf. d. rep's } K)$$

$$M \mapsto \omega M$$

\hookrightarrow an equivalence of categories.

Notation: $G(T, \omega) := \underline{\text{Aut}}^{\otimes}(\omega)$ fund. group
of T wrt ω

Given $M \in \text{ob}(T)$

$\langle M \rangle^{\otimes} :=$ Tannakian subcat. generated
by M , i.e. the smallest
full Tannakian subcat. of T
containing M and closed under
subobjects

$$G(M, \omega) := G(\langle M \rangle^{\otimes}, \omega|_{\langle M \rangle^{\otimes}})$$

fundamental group of M wrt ω

Example: $T = \text{MM}(k)$ mixed motives
 $/K \subset k$

$\omega_B : T \rightarrow \text{Mod}_{\mathbb{Q}}$ Betti realizations

$M \in \text{ob}(T)$

$G(M, \omega_B) =: G^{\text{mot}}(M)$

§ 2.1 From now on, assume:

T a Tannakian cat. $/K$, $\text{char } k = 0$

equipped with a weight filtration W .

similar to \mathfrak{w} the wt filtration
on motives:

H_M , W_M finite filtration
on M

W , functorial, exact,
compatible with \otimes ,

Gr^W exact and faithful

Let $M \in \text{ob}(CT)$.

$$\langle \text{Gr}^W M \rangle^\otimes \subset \langle M \rangle^\otimes \rightsquigarrow G(M, \omega) \xrightarrow{\text{res}} G(\text{Gr}^W M, \omega)$$

$$U(M, \omega) := \ker(\text{res})$$

unipotent group
(unipotent radical of
 $G(M, \omega)$ if $\text{Gr}^W M$ is
semisimple).

$$G(M, \omega) \hookrightarrow \text{Lie } U(M, \omega) =: u(M, \omega)$$

(restriction of the Adjoint rep')

By the fundamental theorem of Tannakian categories, get

$$\underline{u}(M) \in \text{ob} \langle M \rangle^\otimes$$

$$\text{with } \omega \underline{u}(M) = u(M, \omega).$$

(Actually $\underline{u}(M)$ indep. of ω)

Restricting to action on ωM gives

$$G(M, \omega) \hookrightarrow GL(\omega M)$$

This makes

$$\underline{u}(M, \omega) \subset W_{-1} \underline{\text{End}}(\omega M) :=$$
$$\left\{ f \in \underline{\text{End}}(\omega M) : f(\omega W_n M) \subset \omega W_{n-1} M \right\}$$

And

$$\underline{u}(M) \subset W_{-1} \underline{\text{End}}(M)$$
$$(\underline{\text{End}}(M)) = \underline{\text{Hom}}(M, M)$$

§ 3 Deligne's characterization of $\underline{\mathcal{C}}(M)$

$$\mathrm{Ext}(M/W_p M, W_p M) \cong \mathrm{Ext}(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_p M, W_p M))$$

$$0 \rightarrow W_p M \rightarrow M \rightarrow M/W_p M \rightarrow 0 \rightsquigarrow \underbrace{\mathcal{E}_p(M)}_{\downarrow} \\ p\text{-on extensions class of } M$$

$$\underline{\mathrm{Hom}}(M/W_p M, W_p M) \hookrightarrow W_{-1} \underline{\mathrm{End}}(M)$$

↗

$$\mathrm{Ext}(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_p M, W_p M)) \hookrightarrow \mathrm{Ext}(\mathbb{1}, W_{-1} \underline{\mathrm{End}}(M))$$

\downarrow

inj. by wt
considerations

$$\mathcal{E}(M) := \sum_p \mathcal{E}_p(M) \in \mathrm{Ext}(\mathbb{1}, W_{-1} \underline{\mathrm{End}}(M))$$

total ext. class p
of M

Thm (Deligne) : $\underline{u}(M)$ is the smallest
subobject of $W_{\underline{1}} \underline{\text{End}}(M)$ s.t. $\Sigma(M)$
is in the image of the pushforward map

$$\text{Ext}(\underline{1}, \underline{u}(M)) \longrightarrow \text{Ext}(\underline{1}, W_{\underline{1}} \underline{\text{End}}(M))$$

(under inclusion map).

Deligne explicitly constructs an ext. pushing forward to $\Sigma(M)$

§ 4. Refinements about individual $\Sigma_p(M)$'s

Question : Does each $\Sigma_p(M)$ come
from $\underline{u}(M)$?

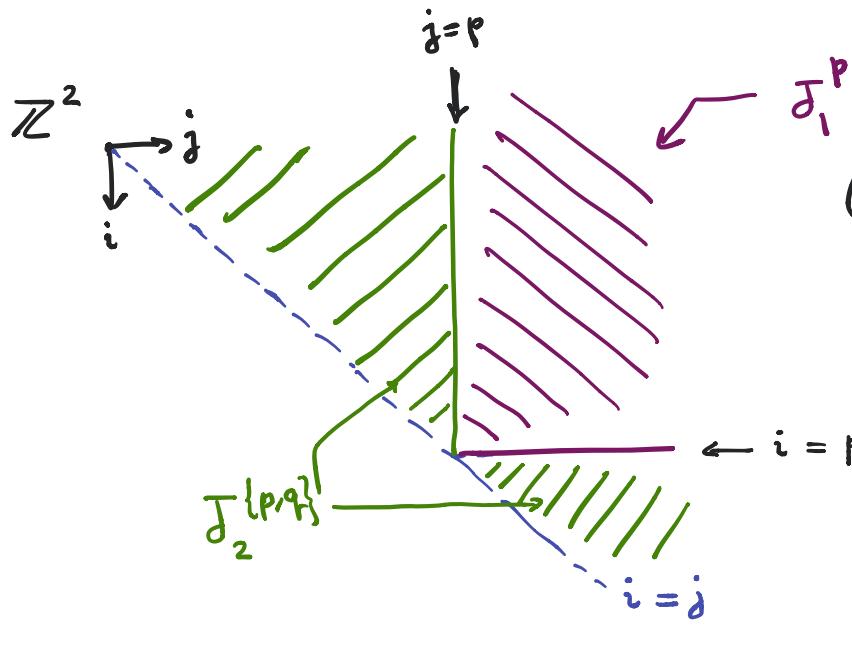
Ans : Not in general.

Easy to construct counterexamples
using Jacquinot - Ribet's deficient
points on 1-motives.

Let

$$\mathcal{J}_p^1 := \{(i, j) \in \mathbb{Z}^2 : i \leq p < j\}$$

$$\mathcal{J}_p^2 := \{(i, j) \in \mathbb{Z}^2 : i < j\} \setminus \mathcal{J}_p^1$$



(Fig. drawn according to labelling of entries of matrices.)

Note: Only depends on $\text{Gr}_j^W M$

Say M satisfies $(IA1)_p$ if

$$\bigoplus_{(i,j) \in \mathcal{J}_p^1} \underline{\text{Hom}}(\text{Gr}_j M, \text{Gr}_i M) \text{ and } \bigoplus_{(i,j) \in \mathcal{J}_p^2} \underline{\text{Hom}}(\text{Gr}_j M, \text{Gr}_i M)$$

have no nonzero isomorphic subobjects.

Example: This holds if the two objects have no common weights, i.e. if

$$\{i-j : (i,j) \in J_1^P ; \text{Gr}_i M, \text{Gr}_j M \neq 0\}$$

and

$$\{i-j : (i,j) \in J_2^P ; \text{Gr}_i M, \text{Gr}_j M \neq 0\}$$

are disjoint.

Thm 1 (E. - K. Murty)

Suppose the following hold :

(1) $\text{Gr}^W M$ is semisimple

(2) M satisfies $(IA1)_p$.

Then $\Sigma_p(M)$ comes from $\underline{\mathfrak{U}}(M)$.

Cor : Suppose M satisfies

$(IA2)$: the numbers

$$i-j \quad (i < j ; \text{Gr}_i M, \text{Gr}_j M \neq 0)$$

are all distinct. (I.e. $W, \underline{\text{End}}(M)$ has $\binom{\#\text{arts of } M}{2}$ distinct arts)

Then every $\Sigma_p(M)$ comes from $\underline{u}(M)$.

Example: Say the wts of M are

$$0, -1, -3, -7$$

Then every $\Sigma_p(M)$ comes from $\underline{u}(M)$.

§5 Applications to motives

Suppose $Gr M$ is semisimple. (^{It will be} when dealing with motives.)

Recall $\underline{u}(M) \subset W_{-1} \underline{\text{End}}(M)$

Def'n: Say $\underline{u}(M)$ is large if $\underline{u}(M) = W_{-1} \underline{\text{End}}(M)$

Here's an application of the previous result:

Thm 2 (E.-K. Murty)

Let $p < 0$. Suppose

$$M/\underset{W_p M}{\sim} \mathbb{1}, \text{ and } Gr_p M \neq 0$$

(i.e. the highest two wts are $p, 0$ and $Gr_0^W M \sim \mathbb{1}$).

Suppose moreover that

$$(1) \quad \underline{\mathcal{L}}(W_p M) \text{ is large}$$

$$(2) \quad \underline{\mathcal{L}}\left(\frac{M}{W_{p-1} M}\right) \text{ is large.}$$

(3) M satisfies $(IA1)_p$.

Then $\underline{\mathcal{L}}(M)$ is large.

Important. Again counterexample with 1-motives

Question: Let $p < 0$.

Given $\begin{cases} L \text{ with } W_p L = L, \text{Gr}_p^W L \neq 0 \\ N \text{ an ext. of } 1 \text{ by } \text{Gr}_p^W L, \end{cases}$

is there an M s.t.

$$\begin{cases} W_p M \simeq L \\ M/\frac{}{W_{p-1} M} \simeq N \end{cases} ?$$

Ans: $\mathcal{L} : \circ \rightarrow W_{p-1} L \rightarrow L \rightarrow G_{p-1} L \rightarrow \circ$

\rightsquigarrow

$$\dots \rightarrow \text{Ext}(\mathbb{1}, W_{p-1} L) \xrightarrow{\mathcal{L} \circ -} \text{Ext}(\mathbb{1}, L) \rightarrow \text{Ext}(\mathbb{1}, G_p L) \xrightarrow{\text{Ext}^2(\mathbb{1}, W_{p-1} L)} \dots$$

\Downarrow
 N
"

$$\circ \rightarrow G_p L \rightarrow N \rightarrow \mathbb{1} \rightarrow \circ$$

Thus M exists iff $\mathcal{L} \circ N = \circ$.

Example 1) Take $T = MT(\mathbb{Q})$

Voevodsky's
mixed Tate
motives / \mathbb{Q}

Recall:

- $\text{Ext}(\mathbb{1}, \mathbb{Q}(n)) = \begin{cases} \mathbb{Q}^* \otimes \mathbb{Q} & n=1 \\ \mathbb{Q} & n=\text{odd}, \geq 3 \\ 0 & \text{otherwise} \end{cases}$
- Higher Ext groups all vanish.

$L = L_3 :=$ a nontrivial ext. of $\mathbb{1}$ by $\mathbb{Q}(3)$

"motive of $\zeta(3)$ "

$\zeta(3)$ is a period

$$N := \left[\mathbb{Z} \xrightarrow{1 \mapsto 2} G_m \right]$$

\hookrightarrow nontrivial ext. of \mathbb{I} by $\mathbb{Q}(1)$

\Rightarrow Can "patch together" $\underbrace{L(1)}$ and N :

\downarrow
nontriv. ext. of
 $\mathbb{Q}(1)$ by $\mathbb{Q}(4)$

unique up to
isomorphism

There is M in $MT(\mathbb{Q})$ with associated
graded $\mathbb{Q}(4) + \mathbb{Q}(1) + \mathbb{I}$, and

$$\left\{ \begin{array}{l} W_{-2}M \simeq L(1) \\ M/W_{-8}M \simeq N. \end{array} \right.$$

By previous result, $\underline{u}(M)$ is large.

$$\begin{aligned} \dim G^{\text{not}}(M) &= \dim \underbrace{u(M, \omega_B)}_{W_{-1}\text{End}_B(\omega M)} \oplus \dim \underbrace{G(\mathbb{Q}(1))}_{G_m} \\ &= 3 + 1 \end{aligned}$$

Period conj \leftrightarrow tr. deg. of field
gen. / \mathbb{Q} by periods of
 M should be 4.

The field gen. by periods includes
 $2\pi i$, $\zeta(3)$, $\log 2 \xrightarrow{\text{coming from}} L(1)$ and N

What is the new period of M ?

(Note: M is not in $MT(\mathbb{Z})$.)

Example 2: Build on previous example.

M as above.

Patch together $M(5)$ and motive L_5

of $\zeta(5)$: get M' with associated
graded

$$\mathbb{Q}(9) + \mathbb{Q}(6) + \mathbb{Q}(5) + \mathbb{L}$$

wts $-18, -12, -10, 0$

and

$$W_{-10}M' \simeq M(5)$$

$$M'/W_{-12}M' \simeq L_5$$

$\underline{u}(M')$ is large. (Note: wts satisfy (IAz).)

$$\dim G^{\text{not}}(M') = \binom{4}{2} + 1 = 7.$$

Field gen. by periods includes

$2\pi i, \zeta^{(3)}, \log 2$, new period of $M, \zeta^{(5)}$.

So M' should have 2 new periods
(i.e. periods not generated by the
previous ones).

§ 6. On the proof of Thm 1

Natural to expect that $\underline{\epsilon}_p(M)$ is related to the subobject

$$\underline{u}_p(M) := \underline{u}(M) \cap \underline{\text{Hom}}(M/W_p M, W_p M).$$

This is indeed the case: easy to prove that

Prop 3: Write

$$\underline{\epsilon}_p(M): 0 \rightarrow \underline{\text{Hom}}(M/W_p M, W_p M) \rightarrow M' \rightarrow \mathbb{I} \rightarrow 0.$$

Then $\underline{u}_p(M)$ is the smallest subobj of

$$\underline{\text{Hom}}(M/W_p M, W_p M) \quad \text{s.t.}$$

$$M' / \underline{u}_p(M) \in \text{Ob} \left\langle W_p M, M/W_p M \right\rangle^{\otimes}.$$

Pf: Take a fiber functor ω .

$$\omega \underline{u}_p(M) = \omega \underline{u}(M) \cap \underline{\text{Hom}}\left(\frac{\omega M}{\omega W_p M}, \omega W_p M\right)$$

is the Lie algebra of the kernel of
 $G(M, \omega) \xrightarrow{\text{restrictions}} G\left(W_p^{M \oplus M} / W_p^M, \omega\right)$.

call this kernel $U_p(M, \omega)$.

An explicit calculation shows that

$$\forall A \subset \underline{\text{Hom}}(M / W_p M, W_p M),$$

$U_p(M, \omega)$ acts trivially

$$\iff u_p(M) \subset A.$$

on $\omega M' / \omega A$

□

Idea of Proof of Thm 1 :

- ① Want to refine Prop. 3 so that
 $\langle W_p M, M / W_p M \rangle^\otimes$ is replaced by smaller categories.

② Need a more natural formulation of Prop 3.

Given $A \in \text{ob}(T)$, $\varepsilon \in \text{Ext}(\mathbb{1}, A)$
 say ε originates from a full Tan.

subcat. S if ε is in the image of

$$\text{Ext}_S(\mathbb{1}, A') \xrightarrow{f^*} \text{Ext}_T(\mathbb{1}, A)$$

for some $A' \in \text{ob}(S)$, $A' \xrightarrow{f} A$.

Reformulation of Prop 3: $\underline{u}_p(M)$ is the
 smallest subobject of $\underline{\text{Hom}}(M/W_p M, W_p M)$ s.t.

$$\varepsilon_p(M)/\underline{u}_p(M)$$

(= push forward of $\varepsilon_p(M)$ under quotient map)

originates from $\langle W_p M, M/W_p M \rangle^\otimes$.

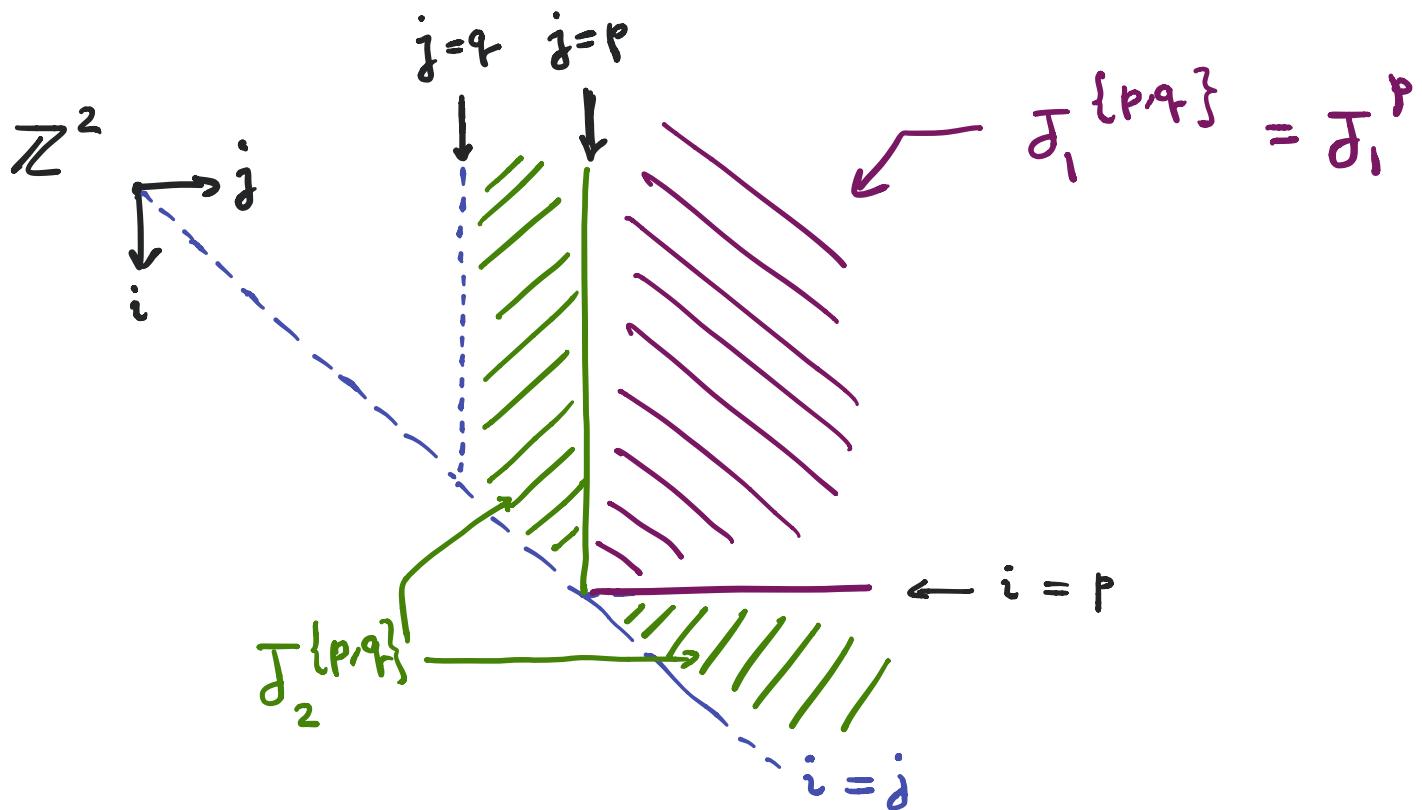
Nice thing: Here $\varepsilon_p(M)/\underline{u}_p(M)$ may be

considered as an element of

$$\mathrm{Ext}(\mathbb{1}, \frac{\underline{\mathrm{Hom}}(M/W_p M, W_p M)}{\underline{u}_p(M)}) \text{ or } \mathrm{Ext}(\mathbb{1}, \frac{W_p \underline{\mathrm{End}}(M)}{\underline{u}_p(M)}).$$

③ Now Prop. 3 can be refined.

let $q \leq p$. Consider $\mathcal{J}_1^{\{p,q\}}$ and $\mathcal{J}_2^{\{p,q\}}$ below.



We actually prove the following:

Thm 3 (E. - K. Murty)

Suppose one of the following holds:

(1) $\text{Gr}^W M$ is semisimple, and the two objects

$\oplus \underline{\text{Hom}}(\text{Gr}_j M, \text{Gr}_i M)$ and $\oplus \underline{\text{Hom}}(\text{Gr}_j M, \text{Gr}_i M)$

$$a_{ij} \in J_1 \{p, q\}$$

$$a_{ij} \in J_2 \{p, q\}$$

have no nonzero isomorphic subobjects.

(2) The two objects above have no common wts.

Then the extension $E_p(M) / \underline{u}_{p(M)}$ originates

from $\underbrace{\langle W_q M, \text{Gr}^W M \rangle^\otimes}$.

\hookrightarrow smaller than $\langle W_p M, \frac{M}{W_p M} \rangle^\otimes$

Indications on the proof:

1)

$$\bigcup_{\geq q} (M, \omega) := \ker(G(M, \omega) \longrightarrow G(W_q^W M \oplus \text{Gr}^W M, \omega))$$

Originating from $\langle W_q M, \text{Gr}^W M \rangle^\otimes$

\braceleftarrow related to

right exactness of $U_{\geq q}(M, \omega)$

invariance applied to the sequence.

Set

$$\underline{\mathcal{U}}_{\geq q}(M) := \underline{\mathcal{U}}(M) \cap \underline{\text{Hom}}(M/W_q M, M)$$

$$\text{Then } \omega \underline{\mathcal{U}}_{\geq q}(M) = \text{Lie } U_{\geq q}(M, \omega)$$

$\underline{\text{H}\omega}$.
 \equiv

2) Trick: Take ω to be a graded

fiber functor: start with arbitrary

fiber functor ω_0 , and take

$$T \xrightarrow{\text{Gr}^W} T \xrightarrow{\omega_0} \text{Mod}_K .$$

$$\omega \quad \omega N = \omega_0 \text{Gr}^W N$$

3) Conditions (1) or (2) of the statement of the theorem imply that $\text{Gr}^W \underline{\omega}_{\geq q}(M)$ decomposes

as

$$\bigoplus_{l=1,2} \left(\text{Gr}^W \underline{\omega}_{\geq q}(M) \cap \bigoplus_{\substack{(i,j) \in J_l \\ l}} \underline{\text{Hom}}(\text{Gr}_i M, \text{Gr}_j M) \right).$$

Hence $\omega \underline{\omega}_{\geq q}(M)$ decomposes accordingly.

The rest is straightforward.