Disk Packing Rigidity and the Second Most Uniform Triangulated Packing Problem

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Review on Tensegrities

A tensegrity framework $G(p)$ in d-space is a complete graph $G = \{V, E = E_- \sqcup E_+ \sqcup E_o \sqcup E_\emptyset\}$, and a configuration $p \in \mathbb{R}^{nd}$ of the graph. Elements of $E_-, E_+, E_o, E_\emptyset$ are called cables, struts, bars, and free edges. Each p_i is a vertex in $\mathbb{R}^d$.

In the rigidity problem of a given configuration p, bars have fixed length, cables cannot increase in length, struts cannot decrease in length, and free edges have no conditions on their length. We say $G(p) \geq G(q)$ if q satisfies all length conditions.

An analytic path $p(t)$ with $p(0) = p$ and $G(p(t)) \leq G(p)$ for $t \in [0, 1]$ is called a motion. We say the p is **rigid** if any motion of p is entirely made of congruent configurations.

Infinitesimal Rigidity

A infinitesimal motion p' is the derivative of a motion at $t = 0$. One can show it is equivalent to a vector such that

$$\begin{aligned}(p_j - p_i) \cdot (p'_j - p'_i) \leq 0 \text{ for } (i,j) \in E_- \\ (p_j - p_i) \cdot (p'_j - p'_i) = 0 \text{ for } (i,j) \in E_o \\ (p_j - p_i) \cdot (p'_j - p'_i) \geq 0 \text{ for } (i,j) \in E_+\end{aligned}$$

If any infinitesimal motion is the derivative of a trivial motion, we say the framework is infinitesimally rigid.

Infinitesimal rigidity implies rigidity(why?)

Rigidity Matrix

The main tool to study infinitesimal rigidity is the rigidity matrix $R(p) \in \mathbb{R}^{|E| \times nd}$ defined as following:

$$\begin{array}{cc} \begin{array}{ccccc} x_i & y_i & \dots & x_j & y_j \end{array} & \\ \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ x_i - x_j & y_i - y_j & 0 & x_j - x_i & y_j - y_i \end{bmatrix} & \begin{array}{c} \dots \\ (i,j) \end{array} \end{array}$$

Here, free edges are ignored because they do not contribute to any constraints.
Observe $\omega R(p)$ gives the vector sum $\sum_j \omega_{ij}(p_i - p_j)$ for each vertex, and $R(p)p'$ gives the dot product $(p_i - p_j) \cdot (p'_i - p'_j)$ for each edge (i,j).
The kernal of $R(p)$ are infinitesimal motions that preserve all lengths, while the cokernal consists of self-stress (as force density per length) that gives 0 net force on each vertex.

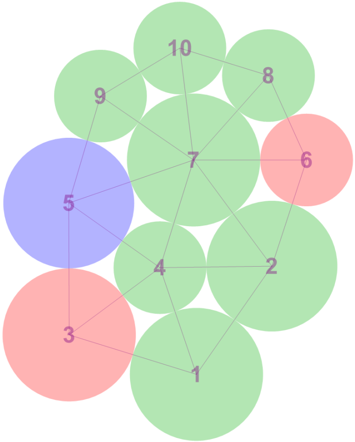
Theorem(Roth-Whiteley): A tensegrity framework is infinitesimal rigid if and only if two conditions hold:
i. it's infinitesimally rigid when all cables and struts have fixed length
ii. it's possible to assign a force density for each edge so that the assigned number is positive on cables, negative on struts, zero on free edges, and arbitrary for bars, so that every vertex has 0 net force.

Disk Packing Rigidity

Given a planar graph $G = (V = V_- \sqcup V_+ \sqcup V_o \sqcup V_\emptyset, E)$ and configuration $p = (x_1, y_1, r_1, ...) \in \mathbb{R}^{3|V|}$, we say $G(p)$ is a packing of G if for each edge $(i,j) \in E$, $(x_i - x_j)^2 + (y_i + y_j)^2 = (r_i + r_j)^2$ and the orientation of each triangle is preserved from underlying complex. Disks in $V-$ are only allowed to decrease their radii, disks in V_+ are only allowed to increase their radii, disks in V_o are fixed in radii, and disks in $V_\emptyset$ are free to change their radii. We say $G(p) \geq G(q)$ if q satisfies all constraints given packing $G(p)$.

A motion of $G(p)$ is an analytic path $G(p(t))$ of packings, $t \in [0,1]$, such that $p(0) = p$ and $G(p(t)) \leq G(p)$ for all t. We say $G(p)$ is a rigid packing if any motion is made of congruent packings.

A Motion



The animation on the left shows a motion of the given packing.

Throughout this presentation, we will label fixed disks green, growing disks red, shrinking disks blue, and free disks gray.

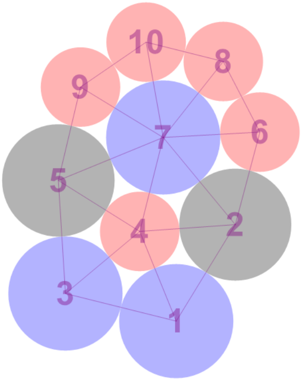
Infinitesimal Rigidity

We can define infinitesimal motion as the derivative of a motion at $t = 0$. It is easy to see an infinitesimal motion is a vector $p' = (x', y', r')$ such that

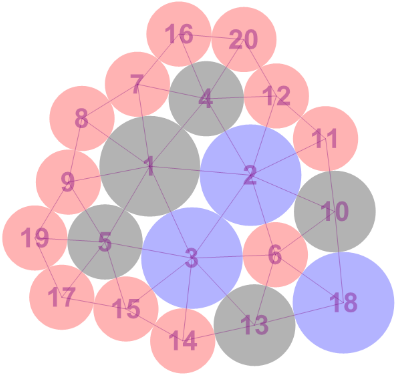
$$(p_j - p_i) \cdot (p'_j - p'_i) = (r_i + r_j)(r'_i + r'_j) \quad \forall (i,j) \in E$$
$$r'_i \geq 0 \quad \forall i \in E_+$$
$$r'_i \leq 0 \quad \forall i \in E_-$$
$$r'_i = 0 \quad \forall i \in E_o$$

We say the packing is infinitesimally rigid if any infinitesimal motion is yielded from a trivial motion.

Rigid packing but not infinitesimally rigid?



This packing is rigid, but not infinitesimally rigid. Can you eyeball an infinitesimal motion?



This packing is both rigid and infinitesimally rigid.

Rigidity Matrix

The equation from last page motivates the following rigidity matrix $R(p)$:

$$\begin{array}{cc} \begin{array}{ccccccc} x_i & y_i & r_i & \dots & x_j & y_j & r_j \end{array} & \\ \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_i - x_j & y_i - y_j & -r_i - r_j & 0 & x_j - x_i & y_j - y_i & -r_i - r_j \end{bmatrix} & \begin{array}{c} \dots \\ (i,j) \end{array} \end{array}$$

One can check $R(p)p'$ yields $(p_j - p_i) \cdot (p'_j - p'_i) - (r_i + r_j)(r'_i + r'_j)$ for each edge. Thus the kernal of the matrix are infinitesimal motions when we assume all disks are free. $\omega R(p)$ yields $\sum_j w_{ij}(p_i - p_j)$ and $\sum_j w_{ij}(-r_i - r_j)$, hence the cokernal can be interpreted a force density on each edge so that each disk has 0 net force and 0 inward-outward force.

In general, there is no reason to expect the cokernal to be nonempty. Intuitively, we should expect disks that cannot grow in size being able to hold a net outward force without causing it to deform. Similarly, disks cannot shrink should be able to hold a net inward force, and fixed disks can hold any radial force. Hence disks that are not free should be allowed to have nonzero net radial force.

Therefore, we can relax the cokernal by extending the matrix with additional rows. For each disk i that is not free, we add a row with 1 on the r_i entry and 0 everywhere else. Denote this extended matrix as $R_e(p)$.

Analog to Roth Whiteley

Theorem(Zhang, Connelly): A circle packing is infinitesimally rigid if and only if the following conditions hold:
i. The packing is infinitesimally rigid when all disks not free are fixed.
ii. There exists a force density on edges such that the net force is 0 in on each vertex, and the radial force balance is positive on shrinking disks, negative on growing disks, and 0 on free disks.

Intuitively, the first condition wants to rule out infinitesimal motions that fix all radius with a constraint. This can be done quickly through computer by computing the dimension of $Ker(R_e(p))$. The second condition kills all infinitesimal motions with radius change in the "right" direction.

Proof Sketch 1

$\leftarrow$: define $w_k = \sum_j w_{kj}(r_k + r_j)$, the radial net force. Then $w = (w_{i,j}, ..., w_k)$ is in the cokernal of $R_e(p)$, where (i,j) are indexed over edges and k is indexed over disks that are not free.

We have

$$wR_e(p)p' = \sum_{(i,j)} w_{ij}((p_i - p_j)\cdot(p'_i - p'_j) - (r_i + r_j)(r'_i + r'_j)) + \sum_{k \notin V_\emptyset} w_k r'_k$$

The first part is 0 by definition of infinitesimal motion. The second term is none-positive because we assumed w_k and r'_k have opposite signs. Hence any $w_k r'_k \neq 0$ would imply $wR_e(p)p' < 0$, contradicting $w \in CoKer(R_e(p))$. Thus, all $r'_k = 0$, and by (i), the packing is infinitesimally rigid.

Proof Sketch 2

$\rightarrow$: If a packing is infinitesimally rigid, then (i) is automatically true. To prove (ii), we employ Fredholm Alternative:
Lemma(Fredholm alternative): Either $Ax = b$ has a solution, or there exists y such that $A^T y = 0$ but $y^T b \neq 0$.

Let $A = R_e(p)^T$ in the above lemma. Let b be any number assigned to a free disk as a growth/shrink factor. The lemma above says either there's an infinitesimal motion fixing all non-free disks that strictly modifies the free disk, or an internal force can prevent the free disk from deforming. Since our packing is infinitesimally rigid, the only possibility is the latter case. Hence any attempt for a free disk to deform without affecting constrained disks is futile.

Proof Sketch 3

Now we have proved there's an internal force balance to counter any grow/shrink factor on free disks. If we can prove there's an internal force balance with correct signs on edges to kill growth/shrink factor on growing/shrinking disks, then taking linear combination of the two will finish the proof.

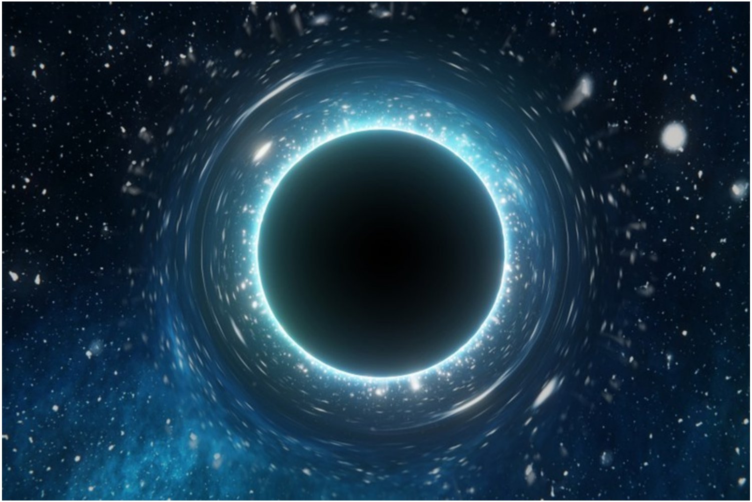
We can simply multiplying a very large number to the later internal force to guarantee all signs are correct in the sum. Next we employ the following lemma:

Proof Sketch 4

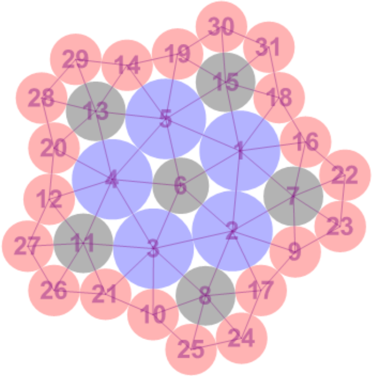
Lemma(Farkas Alternative): Either $Ax = b$ has a solution and $x \geq 0$, or there exists y such that $A^T y \leq 0$ but $y^T b > 0$.

Let $A = R_e(p)^T$ in the above lemma with signs modified on rows. Let b be any numbers assigned to growth/shrink disks. The lemma above says either there's an infinitesimal motion moving a growth/shrink disk in the correct direction, or there exists an internal force with correct signs to counter radial forces of growth/shrink disks. Since our configuration is infinitesimally rigid, only the latter case can be true. $\square$

Global Rigidity

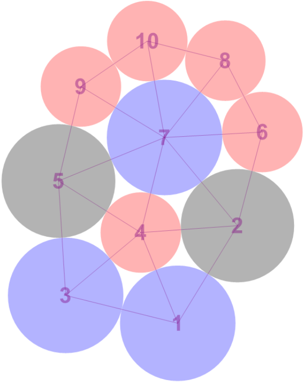


Cases Global Rigidity is known



For this figure on the left, we want to grow all disks on the boundary while shrink disk 1 to 5 in the interior. This cannot be done because it is known interior radii are uniquely determined by boundary radii, and all $\frac{\partial r_{interior}}{\partial r_{boundary}} > 0$

Cases Global Rigidity is known 2



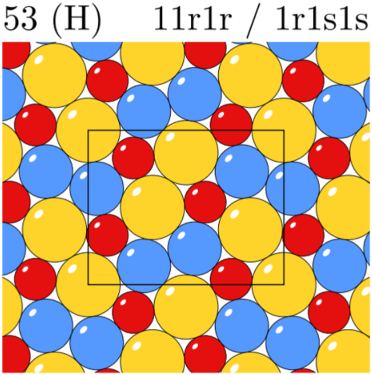
Based on the argument from previous slide, disk 2 and disk 5 cannot both grow or shrink. WLOG we assume disk 5 is shrinking.

We can compute a lower bound for the size of disk 2 given disk 5 by assuming disk 1,3,4,7 are fixed. Similarly, we can compute an upper bound bu fixing disk 4,6,7,8,9,10. The lower bound is greater than the upper bound and they intersect at the configuration on the left.

Why global rigidity matters?

1. If we can determine global rigidity for disk packings, then we can assign the minimal disks to grow and maximal disks to shrink. This would tell us how uniform (in the sense of radii ratio) packings can be given a finite graph.
2. If we can show sufficiently many finite graphs cannot be too uniform, we have insights into what the graph should look like if we want our packing to be uniform. This idea is similar to how four-color theorem is proved.

Why global rigidity matters?



Conjecture (Zhang, Connelly): The packing on the left (found by Fernique) has the maximum $\frac{r_{min}}{r_{max}}$ for any plane triangulated packing that is not hexagonal packing.

I have a collection of approximated packings found through interior point optimization. If they are globally rigid, then the conjecture is true.

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- B. Roth and W. Whiteley, "Tensegrity Frameworks," *Transactions of The American Mathematical Monthly*, vol. 265, no. 2, pp. 419–446, 1991.
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- T. Fernique, A. Hashemi, and O. Sizova, "Compact packings of the plane with three sizes of discs," *Discrete & Computational Geometry*, Jan. 2020.