Coordinate shadows of semi-definite and Euclidean distance matrices
by D. Drusvyatskiy, G. Pataki and H. Wolkowicz

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Linear image of closed convex cones

Linear image of closed convex cones

- Let $\mathcal{A} : \mathbb{E} \to \mathbb{Y}$ be a linear mapping and $C \subset \mathbb{E}$ a closed convex cone
- A central question in convex analysis:

  Is the linear image $\mathcal{A}(C)$ closed?

- This question is connected with:
  1. preservation of lower semi-continuity[1]
  2. uniform duality in conic linear systems[2]
  3. The existence of solutions to extremum problems
- Relation to the "nice cones" [3]

[1] R.T. Rockafellar, Convex Analysis, Princeton Math. Ser. 28, Princeton University Press, Princeton, NJ, 1970.

[2] Duffin, R. J., R. G. Jeroslow, L. A. Karlovitz. Duality in semi-infinite linear programming. Semi-Infinite Programming and Applications (Austin, TX, 1981). Lecture Notes in Econom. and Math. Systems, Vol. 215. Springer, Berlin, Germany, 50-62.

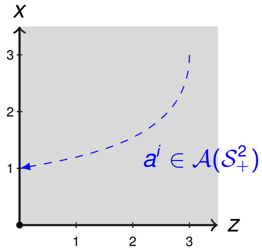
[3] G. Pataki, On the closedness of the linear image of a closed convex cone, Math. Oper. Res., 32 (2007), pp. 395–412.

Linear image of closed convex cones

- Define the mapping $\mathcal{A} : \mathcal{S}^2_+ \to \mathbb{R}^2$ to be

$$\mathcal{A}(X) = \begin{pmatrix} x \\ z \end{pmatrix}, \text{ where } X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0$$

- The image set $\mathcal{A}(\mathcal{S}^2_+)$ is $\mathbb{R}^2$



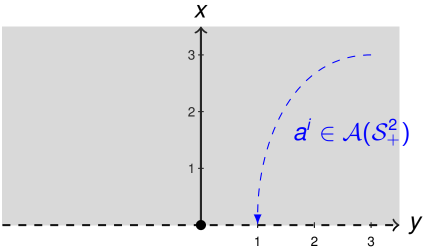
- The image set is closed

Linear image of closed convex cones

- Define the mapping $\mathcal{A} : \mathcal{S}^2_+ \to \mathbb{R}^2$ to be

$$\mathcal{A}(X) = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0$$

- The image set $\mathcal{A}(\mathcal{S}^2_+)$ is $\{(0,0)\} \cup (\mathbb{R}_{++}, \mathbb{R})$



- The image set is NOT closed

The Positive Semidefinite (PSD) Completion Problem

The Positive Semidefinite (PSD) Completion Problem

- Let $G = (V, E)$ be an undirected graph with n vertices
- For $a \in \mathbb{R}^E$, we try to find a p.s.d. $X \in \mathcal{S}^n_+$ such that $X_{ij} = a_{ij}$
- For example, can we find values for the free entries so that $X \succeq 0$ below

$$X = \begin{bmatrix} 7 & 4 & ? & ? \\ 4 & 3 & 5 & ? \\ ? & 5 & ? & 2 \\ ? & ? & 2 & ? \end{bmatrix}$$

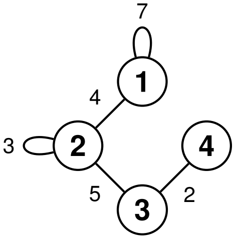


Figure: The associated graph G and $a \in \mathbb{R}^E$

- If we can find a completion, then we call $a \in \mathbb{R}^E$ p.s.d. completable

The Positive Semidefinite (PSD) Completion Problem

- Define the mapping $\mathcal{P} : \mathcal{S}^n_+ \to \mathbb{R}^E$ to be

$$\mathcal{P}(X) = (X_{ij})_{ij \in E}$$

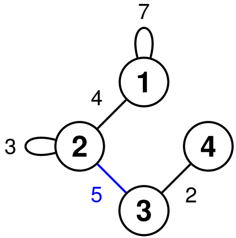
- $\mathcal{P}(S^n_+) \subset \mathbb{R}^E$ is the set of p.s.d. completable vectors

Theorem (D. Drusvyatskiy, G. Pataki and H. Wolkowicz)

The projected set $\mathcal{P}(\mathcal{S}^n_+)$ is closed if and only if L and L^c are disconnected, where $L := \{i : (i,i) \in E\}$.

- $L = \{1, 2\}$ and $L^c = \{3, 4\}$ in this example. So $\mathcal{P}(\mathcal{S}^n_+)$ is not closed

$$X = \begin{bmatrix} 7 & 4 & ? & ? \\ 4 & 3 & 5 & ? \\ ? & 5 & ? & 2 \\ ? & ? & 2 & ? \end{bmatrix}$$



The matrix completion problem

- We prove if L is disconnected from L^c, then $\mathcal{P}(\mathcal{S}^n_+)$ is closed
- Let $a^i \in \mathcal{P}(\mathcal{S}^n_+)$ and $a^i \to a \in \mathbb{R}^E$
- $a^i = \mathcal{P}(X^i)$ for some $X^i \in \mathcal{S}^n_+$
- (Special case) Assume $L = \{1, \dots, n\}$, e.g.,

$$X = \begin{bmatrix} 7 & 2 & ? & ? \\ 2 & 3 & ? & ? \\ ? & ? & 1 & ? \\ ? & ? & ? & 2 \end{bmatrix}$$

- As the diagonal elements of X^i converges to some constants, the matrices X^i are bounded
- There exists a convergent subsequence of X^i, say $X^i \to X \succeq 0$
- By the continuity, $\mathcal{P}(X) = a$ and thus $a \in \mathcal{P}(\mathcal{S}^n_+)$
- Thus $\mathcal{P}(\mathcal{S}^n_+)$ is closed

The matrix completion problem

- We prove if L is disconnected from L^c, then $\mathcal{P}(\mathcal{S}^n_+)$ is closed
- (General case) Assume $L = \{1, \dots, r\}$ for some integer $r \geq 0$, e.g.,

$$X = \left[\begin{array}{cc|cc} 7 & 2 & ? & ? \\ 2 & 3 & ? & ? \\ \hline ? & ? & ? & 3 \\ ? & ? & 3 & ? \end{array}\right] \text{ and } r = 2$$

- Applying the special case to the $r \times r$ leading principal minor to obtain the completion $X_L \in \mathcal{S}^r_+$
- The lower-right block is always p.s.d. completable: Let Y be any completion of the restriction of a to L^c. Then

$$\begin{bmatrix} X_L & 0 \\ 0 & Y + \lambda I \end{bmatrix} \succeq 0 \text{ for sufficiently large} \lambda > 0$$

- Thus $\mathcal{P}(\mathcal{S}^n_+)$ is closed

The matrix completion problem

- We prove if L is NOT disconnected from L^c, then $\mathcal{P}(\mathcal{S}^n_+)$ is NOT closed
- (Special case) Assume $n = 2$
- For any $k > 0$,

$$\begin{bmatrix} k^{-1} & 1 \\ 1 & ? \end{bmatrix} \text{ is p.s.d. completable as } X^k = \begin{bmatrix} k^{-1} & 1 \\ 1 & \lambda \end{bmatrix} \succeq 0 \text{ for large } \lambda$$

- Let $a^k = \mathcal{P}(X^k)$. Then $a^k \in \mathcal{P}(\mathcal{S}^2_+)$
- But $a^k \to a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The partial matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$$

  is NOT p.s.d. completable. Thus $a \notin \mathcal{P}(\mathcal{S}^2_+)$ and $\mathcal{P}(\mathcal{S}^n_+)$ is NOT closed
- (General case) Applying the special case to the 2 by 2 submatrix associated to $i \in L$ and $j \in L^c$

Facial Reduction

Strict feasibility

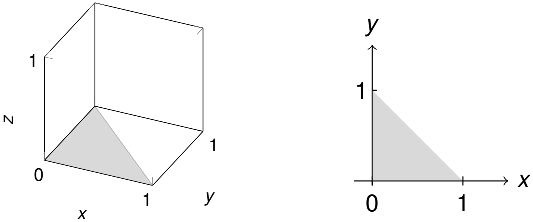
- Let $\mathcal{A} : \mathbb{E} \to \mathbb{Y}$ and $C \subset \mathbb{E}$ a proper closed convex cone. Define the set

$$\mathcal{F} := \{X \in C : \mathcal{A}(X) = b\}$$

- $\mathcal{F}$ is called strictly feasible, if there exists $X \in \mathcal{F} \cap \mathrm{int}(C)$
- Without strict feasibility, optimization over $\mathcal{F}$ may be difficult as
  1. the KKT conditions may not be necessary for the optimality
  2. strong duality may not hold
  3. small perturbations may render the problem infeasible
  4. many solvers might run into numerical errors
- Facial reduction is a regularization technique that can be used for abstract convex programs without strict feasibility (*Borwein and Wolkowicz, 1981*)

Facial Reduction (FR) for Linear Programs

- $\mathcal{F} := \{(x, y, z) \in \mathbb{R}^3_+ : x + y \leq 1, z = 0\}$ is not strictly feasible, as $z = 0$



- Facial reduction yields $\tilde{\mathcal{F}} := \{(x, y) \in \mathbb{R}^2_+ : x + y \leq 1\}$ which is strictly feasible
- Facial reduction removes redundant variables

Facial Reduction (FR) for Semidefinite Programs

- Let $\mathcal{F} := \{X \in \mathcal{S}^n_+ \mid \mathcal{A}(X) = b\}$ be given.
- **(Special case)** Assume that

$$X = \begin{bmatrix} R & \\ & \mathbf{0} \end{bmatrix} \forall X \in \mathcal{F}$$

- This means $X \in \mathcal{S}^n_+ \iff R \in \mathcal{S}^r_+$
- Facial reduction yields an equivalent smaller problem

$$\tilde{\mathcal{F}} := \{R \in \mathcal{S}^r_+ \mid \tilde{\mathcal{A}}(R) = b\}$$

- **(General case)** If strict feasibility fails, then there always exists an orthogonal matrix P such that

$$P^T X P = \begin{bmatrix} R & \\ & \mathbf{0} \end{bmatrix} \forall X \in \mathcal{F}$$

- The key of FR is finding the orthogonal transformation P

Facial reduction for semidefinite programming

- Recall that Slater's condition holds if there exists X such that

$$\mathcal{A}(X) = b,\ X \in \mathcal{S}^n_{++} \tag{1}$$

- In the next picture, X^* satisfies (1) and thus Slater's condition holds

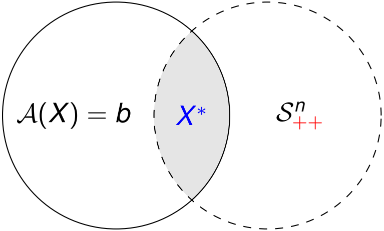


Figure: The intersection is NOT empty

Facial reduction for semidefinite programming

- Recall that Slater's condition holds if there exists X such that

$$\mathcal{A}(X) = b,\ X \in \mathcal{S}^n_{++} \tag{2}$$

- In the next picture, Slater's condition doesn't hold

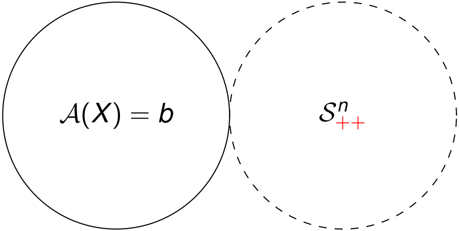
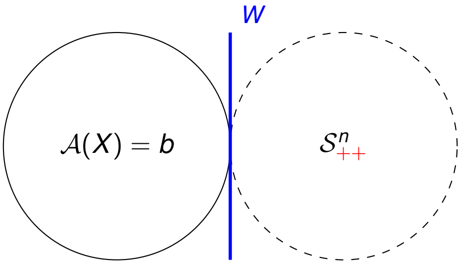


Figure: The intersection is empty

- What is the most important result about two disjoint convex sets? It must be the hyperplane separation theorem

Facial reduction for semidefinite programming

- If Slater's condition fails, then there exists a separating hyperplane W



- The orthogonal transformation P can be obtained from W easily
- W is called an exposing vector and it can be obtained by solving the auxiliary system

$$0 \neq \mathcal{A}^*(v) \in \mathcal{S}^n_+ \text{ and } \langle v, b \rangle = 0$$

  and setting $W = \mathcal{A}^*(v)$

Facial reduction for semidefinite programming

Theorem (J. Borwein and H. Wolkowicz)

Let

$$\mathcal{F} := \{X \in \mathcal{S}^n_+ : \mathcal{A}(X) = b\}$$

be given. Then exactly one of the following statements holds.

- 1. $\mathcal{F}$ *is strictly feasible*
  2. *There exists a vector* v *such that*

$$0 \neq \mathcal{A}^*(v) \in \mathcal{S}^n_+ \text{ and } \langle v, b \rangle = 0$$

- The exposing vector $W = \mathcal{A}^*(v)$ yields a smaller problem

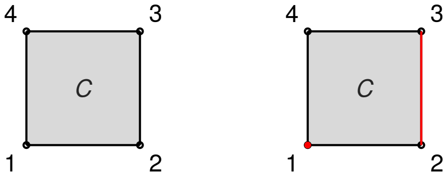
$$\tilde{\mathcal{F}} := \{R \in \mathcal{S}^r_+ \mid \tilde{\mathcal{A}}(R) = b\}$$

- If $\tilde{\mathcal{F}}$ is strictly feasible, then we are done
- If not, then we can repeat the facial reduction algorithm

Facial reduction and exposed faces

Faces of convex sets

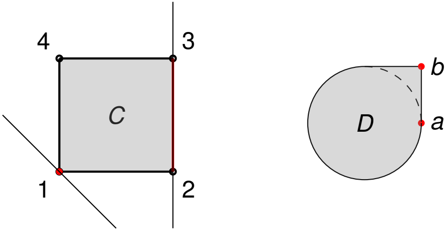
- Let C be a convex set. A convex set $F \subseteq C$ is called a *face* of C if for every $x \in F$ and $y, z \in C$ such that $x \in (y, z)$, we have $y, z \in F$.



- For example, the vertex $\{1\}$ is a face
- The edge $[2, 3]$ is a face
- If $F \neq \emptyset$ and $F \neq C$, then F is proper

Exposed Faces of convex sets

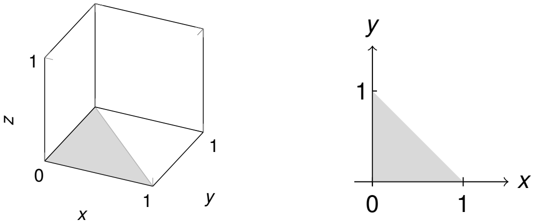
- We say a face $F \subseteq C$ is *exposed* if there exists a supporting hyperplane H to the set C such that $F = C \cap H$



- The faces $\{1\}$ and $[2,3]$ of C are exposed faces
- The face $\{a\}$ of D is NOT exposed, the only supporting hyperplane containing $\{a\}$ includes $[a, b]$

Exposed Faces of convex cones

- Let C be a convex cone. A face F of C is an exposed face when there exists a vector $v \in C^*$ satisfying $F = C \cap v^\perp$
- In this case, we say v exposes F
- Recall that $\mathcal{F} := \{(x, y, z) \in \mathbb{R}^3_+ : x + y \leq 1, z = 0\}$ does not satisfy Slater's condition, as $z = 0$



- The vector $v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ exposes the minimal face F of $\mathbb{R}^3_+$ containing $\mathcal{F}$

Facial reduction and exposed faces

Theorem (J. Borwein and H. Wolkowicz)

Let

$$\mathcal{F} := \{X \in C : \mathcal{A}(X) = b\}$$

be given. Then exactly one of the following statements holds.

1. *$\mathcal{F}$ is strictly feasible*
2. *There exists a vector v such that*

$$0 \neq \mathcal{A}^*(v) \in C^* \text{ and } \langle v, b \rangle = 0$$

- The second item in the theorem means

  $W = \mathcal{A}^*(v) \in C^*$ exposes a face F of C containing the feasible set $\mathcal{F}$

Exposed faces of a convex cone C and the image set $\mathcal{A}(C)$

Exposed faces of a convex cone C and the image set $\mathcal{A}(C)$

Theorem (D. Drusvyatskiy, G. Pataki and H. Wolkowicz)

Let

$$\mathcal{F} := \{X \in C : \mathcal{A}(X) = b\}$$

be given. Then a vector v satisfies

$$\mathcal{A}^*(v) \in C^* \text{ exposes a proper face of } C \text{ containing } \mathcal{F}$$

if and only if

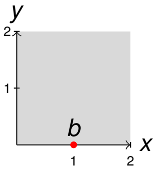
$$v \text{ exposes a proper face of } \mathcal{A}(C) \text{ containing } b.$$

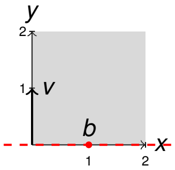
Exposed faces of a convex cone C and the image set $\mathcal{A}(C)$

- Define the mapping $\mathcal{A} : \mathcal{S}^3_+ \to \mathbb{R}^2$ and $b \in \mathbb{R}^2$ to be

$$\mathcal{A}(X) = \begin{pmatrix} X_{11} \\ X_{33} \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- The image set $\mathcal{A}(\mathcal{S}^3_+)$ is $\mathbb{R}^2_+$





- $v = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ exposes a face of $\mathcal{A}(\mathcal{S}^3_+)$ containing b

- By theorem, $\mathcal{A}^*(v) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ exposes a face of $\mathcal{S}^3_+$ containing $\mathcal{F}$

Singularity degree

Theorem (J. Borwein and H. Wolkowicz)

Let

$$\mathcal{F} := \{X \in C : \mathcal{A}(X) = b\}$$

be given. Then exactly one of the following statements holds.

1. $\mathcal{F}$ *is strictly feasible*
2. *There exists a vector* v *such that*

$$0 \neq \mathcal{A}^*(v) \in C^* \text{ and } \langle v, b \rangle = 0$$

- If $\tilde{\mathcal{F}}$ is strictly feasible, then we are done
- If not, then we can repeat the facial reduction algorithm
- (J.F. Sturm) The singularity degree of $\mathcal{F}$ is the smallest number of facial reduction steps needed to obtain a strictly feasible formulation $\tilde{\mathcal{F}}$

Exposed faces of a convex cone C and the image set $\mathcal{A}(C)$

Theorem (D. Drusvyatskiy, G. Pataki and H. Wolkowicz)

Let

$$\mathcal{F} := \{X \in C : \mathcal{A}(X) = b\}$$

be given. Then a vector v satisfies

$$\mathcal{A}^*(v) \in C^* \text{ exposes a proper minimal face of } C \text{ containing } \mathcal{F}$$

if and only if

$$v \text{ exposes a proper minimal face of } \mathcal{A}(C) \text{ containing } b.$$

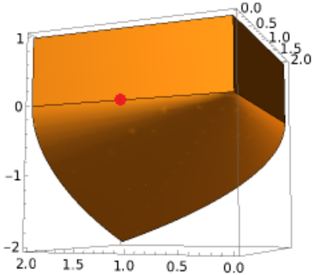
- In this case, the singularity degree is exactly one

Exposed faces of a convex cone C and the image set $\mathcal{A}(C)$

- Define the mapping $\mathcal{A} : \mathcal{S}^3_+ \to \mathbb{R}^2$ and $b \in \mathbb{R}^3$ to be

$$\mathcal{A}(X) = \begin{pmatrix} X_{11} \\ X_{33} \\ X_{22} + X_{13} \end{pmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- The image set $\mathcal{A}(\mathcal{S}^3_+)$ is $R^3_+ \cup \{(x, y, z) : x \geq 0, y \geq 0, xy \geq z^2\}$



- The smallest face of $\mathcal{A}(\mathcal{S}^3_+)$ containing b is not exposed. Thus, the singularity degree is at least two

An application in the PSD completion problem

An application in the PSD completion problem

- Let $G = (V, E)$ be an undirected graph with n vertices
- For $a \in \mathbb{R}^E$, we would like to find a matrix in

$$\mathcal{F} := \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \text{ for all } ij \in E\}$$

- Each clique χ in G yields an exposing vector for $\mathcal{F}$
- For example, the clique $\{1, 2\}$ is associated to the blue submatrix whose rank is one

$$X = \begin{bmatrix} 1 & 1 & ? & ? \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & -1 \\ ? & ? & -1 & 2 \end{bmatrix} \text{ and } \operatorname{face}(\mathcal{F}, \mathcal{S}^4_+) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} S^2_+ \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Theorem (D. Drusvyatskiy, G. Pataki and H. Wolkowicz)

If the subgraph associated to L is chordal, then exposing vectors obtained from all the maximal cliques yield the minimal face of $\mathcal{S}^n_+$ containing $\mathcal{F}$

Thanks for your attention!

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