Euclidean Distance Matrix Completion with One Missing Node

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Single Source localization problem

Given n sensors (towers) and source (cellphone) in dimension r space, assume the locations of the sensors are known and given by

$$P_T = \begin{bmatrix} p^1 & p^2 & \dots & p^n \end{bmatrix}^T \in \mathbb{R}^{n \times r}.$$

and the distance between the source and sensors are contaminated with noise.

$$d_i := \bar{d}_i + \varepsilon_i,\ i = 1, \dots, n,$$

where $\bar{d}_i$ is the true distance and ε_i is a perturbation, or noise.

Three models

1. Fix the the distance between sensors.
2. Allow the sensors to move but can be translated into its original position by an invertible matrix.
3. Allow the sensors to move completely free.

Source Localization Problem

Given n sensors (towers) and source (cellphone) in dimension r space, assume the locations of the sensors are known and given by

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Assumption

The following holds throughout:

1. $n \geq r + 1$;
2. $\mathrm{int}\ \mathrm{conv}(p^1, \dots, p^n) \neq \emptyset$;
3. $\sum_{i=1}^{n} p^i = 0$.

The LS formulation

Using the Euclidean norm as a metric, we obtain the least squares problem

$$p^*_{\mathbf{LS}} := \min_{x\in\mathbb{R}^r} \sum_{i=1}^{n} \left(\|x - p^i\| - d_i\right)^2 . \tag{1}$$

• Iolution is the maximum likelihood estimator when the noise is assumed to be normal and the covariance matrix a multiple of the identity.

• Non-convex and Non-differentiable.

The SLS formulation

The main problem we consider instead is the optimization problem with squared distances

$$\boxed{(\mathbf{SLS}) \quad p^*_{\mathrm{SLS}} := \min_{x\in\mathbb{R}^r} \textstyle\sum_{i=1}^{n} \left(\|x - p^i\|^2 - d_i^2\right)^2 .} \tag{2}$$

GTRS, generalized trust region subproblem

Substitute using $\|x\|^2 = \alpha$,

$$p^*_{\text{SLS}} = \min_{x,\alpha} \left\{ \sum_{i=1}^{n} \left(\alpha - 2x^T p^i + \|p^i\|^2 - d_i^2 \right)^2 : \|x\|^2 - \alpha = 0, \ x \in \mathbb{R}^r \right\}.$$

Standard trust region subproblem. Strong duality is proved in [33, 29] (T.K. Pong, R. Stern and H. Wolkowicz).

Attainment, finiteness and Strong duality

1. The problem **SLS** is equivalent to

   (GTRS) $\quad p^*_{\text{SLS}} = \min\{\|Ay - b\|^2 : y^T \tilde{I} y + 2\tilde{b}^T y = 0, \ y \in \mathbb{R}^{r+1}\}.$

2. $\text{rank}(A) = r + 1$ and the optimum of **GTRS** is finite and attained.
3. Strong duality holds for **GTRS**, dual value is attained:

$$p^*_{\textbf{SLS}} = d^*_{\text{SLS}} := \max_{\lambda} \min_{y} \{\|Ay - b\|^2 + \lambda(y^T \tilde{I} y + 2\tilde{b}^T y)\}. \qquad (3)$$

Note (3) is a dual-form **SDP** corresponding to the primal SDP problem,

$$(\mathbf{SDR}) \qquad \begin{aligned} p^*_{\mathbf{SDR}} := \min\ & \langle \bar{A}, X \rangle \\ \text{s.t.}\ & \langle \bar{B}, X \rangle = 0 \\ & X_{r+2,r+2} = 1, \quad X \in \mathcal{S}^{r+2}_{+}. \end{aligned} \tag{4}$$

Define the map $\rho : \mathbb{R}^{r+1} \to \mathcal{S}^{r+2}$ as,

$$\rho(y) = \begin{pmatrix} y \\ 1 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix}^T. \tag{5}$$

Let Ω denote the optimal set of solutions of **SDR**.

The following holds:

1. The optimal values of **GTRS**, **SDR** are all equal, finite, and attained.
2. The matrix X^* is an extreme point of Ω if, and only if, $y^* = \rho^{-1}(X^*)$ for some minimizer, y^*, of **GTRS**.
3. If **GTRS** has a unique minimizer, say y^*, then the optimal set of **SDR** is the singleton $\rho(y^*)$.
4. If the optimal set of **SDR** is a singleton, say X^*, then $\operatorname{rank}(X^*) = 1$ and $\rho^{-1}(X^*)$ is the unique minimizer of **GTRS**.

Suppose the optimal solution of (4) is $\bar{X}$ and $\text{rank}(\bar{X}) = \bar{r}$ where $\bar{r} > 1$.

$$\bar{X} := UDU^T, \quad D \in \mathcal{S}^{\bar{r}}_{++}.$$

$$\bar{B} \leftarrow U^T \bar{B} U,\ \bar{A} \leftarrow U^T \bar{A} U,\ \bar{E} \leftarrow U^T \bar{E} U, \tag{6}$$

where $\bar{E} := e_{r+2} e_{r+2}^T$.

Define the linear map $\mathcal{A} : \mathcal{S}^{\bar{r}} \to \mathbb{R}^3$ and $b \in \mathbb{R}^3$ as,

$$\mathcal{A}_S(S) := \begin{pmatrix} \langle \bar{B}, S \rangle \\ \langle \bar{A}, S \rangle \\ \langle \bar{E}, S \rangle \end{pmatrix}, \quad b_S := \begin{pmatrix} 0 \\ p^*_{\mathbf{SDR}} \\ 1 \end{pmatrix}, \tag{7}$$

Choose $C \in \text{Null}(\mathcal{A}_S) \setminus \{0\}$, the rank reducing program is

$$\begin{array}{ll} \min & \langle C, S \rangle \\ \text{s.t.} & \mathcal{A}_S(S) = b_S \\ & S \in \mathcal{S}^{\bar{r}}_{+}. \end{array} \tag{8}$$

A purificatin algorithm

Lemma 1

Let $k \geq 1$ be an integer and suppose that C^k, $\mathcal{A}_S^k$, and b_S^k are as in Algorithm 1. Then

$$S^{k+1} \succ 0 \iff \mathcal{F}^k := \left\{S \succeq 0 : \mathcal{A}_S^k(S) = b_S^k\right\} = \left\{S^{k+1}\right\}.$$

Suppose $S^{k+1} \succ 0$. Then

$$X^{k+1} := U^0 \cdots U^k S^{k+1} (U^0 \cdots U^k)^T \in \Omega. \tag{9}$$

is an extreme point of Ω.

Algorithm 1 Purification Algorithm

INPUT: $\mathcal{A}_S$ and $\bar{X} \in \Omega$.
initialize: $k = 1$, $\mathcal{A}_S^1 := \mathcal{A}_S$, $S^1 := \bar{X}$, $U^0 = I$.
while $\mathrm{rank}(S^k) \geq 2$ **do**
  Compute $S^k = U^k D^k (U^k)^T$, with $D^k \in \mathcal{S}_{++}^{r_k}$.
  Redefine $\mathcal{A}_S^k$ and b_S^k using U^k and ensure that it is full rank.
  Choose $C^k \in \mathrm{Null}(\mathcal{A}_S^k) \setminus \{0\}$.
  Obtain $S^{k+1} \in \arg\min\{\langle C^k, S\rangle : \mathcal{A}_S^k(S) = b_S^k,\ S \succeq 0\}$.
  Update $k \leftarrow k + 1$.
end while
OUTPUT: $X^* := U^0 \cdots U^{k-1} S^k (U^0 \cdots U^{k-1})^T$.

Theorem 2

Let $\bar{X} \in \mathcal{S}_+^{r+2}$ *be an optimal solution to* SDR *. If* $\bar{X}$ *is an input to Algorithm 1, then the algorithm terminates with at most* $\text{rank}(\bar{X}) - 1 \leq r + 1$ *calls to the while loop and the output,* X^**, is a rank* 1 *optimal solution of* SDR*.*

Compare with the approach used by Beck et al [3].

$$\begin{array}{rcl} (AA^T + \lambda \tilde{I})y & = & A^T b - \lambda \tilde{b}, \\ y^T \tilde{I} y + 2\tilde{b}^T y & = & 0, \\ A^T A + \lambda \tilde{I} & \succeq & 0. \end{array} \tag{10}$$

1. The so-called *hard case* results in $A^T A + \lambda^* \tilde{I}$ being singular for the optimal λ^* and this can cause numerical difficulties.
2. In our **SDP** relaxation, we need not differentiate between the *'hard case'* and *'easy case'*.

The corresponding **EDM** restricted to the towers is denoted D_T and is defined by

$$(D_T)_{ij} := \|p^i - p^j\|^2, \quad \forall 1 \leq i, j \leq n.$$

Then the approximate **EDM** for the sensors and the source is

$$D_{T_c} := \begin{bmatrix} D_T & d \circ d \\ (d \circ d)^T & 0 \end{bmatrix} \in \S^{n+1}.$$

The nearest **EDM** problem with fixed sensors is

$$\min_{x \in \mathbb{R}^r} \frac{1}{2} \left\| \mathcal{K}\left(\begin{bmatrix} P_T \\ x^T \end{bmatrix} \begin{bmatrix} P_T \\ x^T \end{bmatrix}^T \right) - D_{Tc} \right\|^2. \qquad (11)$$

Relaxation:

$$
(\text{NEDM}) \qquad \begin{array}{rl} \min & \frac{1}{2}\| \mathcal{K}(X) - D_{T_c} \|^2 \\ \text{s.t.} & \operatorname{rank}(X) \le r \\ & X \succeq 0. \end{array} \tag{12}
$$

$X \succeq 0$ in **NEDM** is refined to $X \in \operatorname{face}(F_T, \mathcal{S}_+^{n+1})$:

$$
(\text{NEDMP}) \qquad \begin{array}{rl} \min & \frac{1}{2}\| \mathcal{K}(X) - D_{T_c} \|^2 \\ \text{s.t.} & \operatorname{rank}(X) \le r \\ & X \in \operatorname{face}(F_T, \mathcal{S}_+^{n+1}). \end{array} \tag{13}
$$

The true Gram matrix, $\mathcal{K}^\dagger(\overline{D})$, belongs to the set,

$$
F_T := \{ X \in \mathcal{S}_{c,+}^{n+1} : \mathcal{K}(X)_{1:n,1:n} = D_T \}. \tag{14}
$$

Closed form expression for face$(F_T, \mathcal{S}_+^{n+1})$.

$$G_T =: \begin{bmatrix} U & \frac{1}{\sqrt{n}}e & W_T \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & \frac{1}{\sqrt{n}}e & W_T \end{bmatrix}^T, U^T U = I_r,\ U^T e = 0,\ \Lambda \in \mathcal{S}_{++}^r.$$

$W_T W_T^T$ is an exposing vector for face$(G_T, \mathcal{S}_{c,+}^n)$ since the following hold:

$$\langle G_T, W_T W_T^T \rangle = 0,\ \operatorname{rank}(G_T + W_T W_T^T) = n - 1 = \max_{X \in \mathcal{S}_{c,+}^n} \operatorname{rank}(X).$$

Extend $W_T W_T^T$ to an exposing vector for face$(F_T, \mathcal{S}_+^{n+1})$.

Lemma 3

Let $\overline{W}_T := [W_T^T \quad 0]^T$ *and let* $W := \overline{W}_T \overline{W}_T^T + ee^T$*. Then,*

1. $\overline{W}_T \overline{W}_T^T$ *exposes* face$(F_T, \mathcal{S}_{c,+}^{n+1})$*,*
2. W *exposes* face$(F_T, \mathcal{S}_+^{n+1})$*.*

Define the composite map $\mathcal{K}_V := \mathcal{K}(V \cdot V^T)$ and introduce a weight matrix to the objective,

$$\text{(FNEDM)} \quad \begin{array}{rl} V_\alpha := \min & \frac{1}{2}||H_\alpha \circ (\mathcal{K}_V(R) - D_{T_c})||^2, \qquad (=: f(R,\alpha)) \\ \text{s.t.} & \operatorname{rank} R \le r, \\ & R \succeq 0. \end{array} \tag{15}$$

Here $H_\alpha := \alpha H_T + H_c$ and α is positive.

Theorem 4

Let P_T be as above, $V = \operatorname{Null}(W)$, and let P be a centered matrix with,

$$P = \begin{bmatrix} T \\ c^T \end{bmatrix}, \; T \in \mathbb{R}^{n \times r}, \; c \in \mathbb{R}^r.$$

Then there exists a matrix $Q \in \mathbb{R}^{r \times r}$ such that $P_T Q = J_n T$ if, and only if,

$$PP^T \in V \mathcal{S}_+^{r+1} V^T.$$

The solution to the least squares problem is,

$$R_{LS} := (H_\alpha \circ \mathcal{K}_V)^\dagger (H_\alpha \circ D_{T_c}) \in \operatorname{argmin} f(R). \tag{16}$$

Three cases regarding the eigenvalues of R_{LS},

1. **Case I:** $R_{LS} \succeq 0$ and $\operatorname{rank}(R_{LS}) \leq r$.
2. **Case II:** $R_{LS} \notin \mathcal{S}_+^{r+1}$.
3. **Case III:** $R_{LS} \succ 0$.

Case I and II and be solved by simply dropping out the rank constraint.

In **Case III** motivated by the primal-dual and penalty approach H.D.Qi, X.M. Yuan, G. Sun and D.F. Sun [19, 30, 31].

$$
\text{(PNEDM)} \qquad \begin{aligned} \min \ & \frac{1}{2}||H_\alpha \circ (\mathcal{K}_V(R)) - D_{T_c})||^2 + \gamma p(R), \\ \text{s.t.} \quad & R \succeq 0. \end{aligned} \tag{17}
$$

Algorithm 2 Majorization Algorithm

INPUT: $R_0 \succeq 0$, $\gamma >> 0$, $1 > \epsilon > 0$
initialize: $k = 0$, $err = 1$
while $err > \epsilon$ **do**
  Choose $U^k \in \partial p(R^k)$
  Obtain R^{k+1},

$$
R^{k+1} \in \operatorname*{argmin}_{R \succeq 0} \frac{1}{2}||H_\alpha \circ (\mathcal{K}_V(R)) - D_{T_c})||^2 + \gamma(p(R^k) + \langle U^k, R - R^k \rangle) \tag{18}
$$

  Update $err \leftarrow \|R^{k+1} - R^k\|$, $k \leftarrow k + 1$
end while

Theorem 5

Suppose Algorithm 2 converges to a stationary point $\bar{R}$, and that $\text{rank}(\bar{R}) = r$. *Then $\bar{R}$ is a global minimizer of* FNEDM *restricted to* $\text{face}(\bar{R})$.

Identifying Outliers using l_1 Minimization and Facial Reduction

Using a new notation, problem (15) is equivalent to,

$$\begin{aligned} \min \ & \|\delta\| \\ \text{s.t} \ & Az - b = \delta \\ & \text{s2Mat}(z) \succeq 0 \end{aligned} \tag{19}$$

Consider the popular l_1 norm minimization problem,

$$\begin{aligned} \min \ & \|\delta\|_1 \\ \text{s.t} \ & Az - b = \delta \\ & \text{s2Mat}(z) \succeq 0. \end{aligned} \tag{20}$$

Algorithm 3 Removing Outliers

1: **INPUT:** Matrix of sensor locations, P_T, and vector of noisy distances, d, from sensors to the source.
2: Solve the following l_1 norm minimization problem

$$\begin{array}{l} \min \| \mathcal{K}_V(R) - D_{T_c} \|_1, \\ \text{s.t. } R \succeq 0. \end{array} \tag{21}$$

3: Obtain $\delta := (\mathcal{K}_V(R) - D_{T_c})_{1:n,n+1}$.
4: Normalize: $\delta \leftarrow \frac{1}{\|\delta\|_2}\delta$.
5: Remove p_i from P_T and d_i from d for all i satisfying $\delta_i \geq \frac{1}{\sqrt{n}}$.
6: **OUTPUT:** Sensor matrix P_T and distance vector d with outliers removed.

Recovering Source Position from Gram Matrix

Suppose that the, appropriately partitioned, final **EDM**, corresponding Gram matrix and points are,

$$D_f = \begin{bmatrix} \bar{D}_f & d_f \\ d_f^T & 0 \end{bmatrix}, \; G_f = P_f P_f^T \in \mathcal{S}^{n+1}, \quad P_f = \begin{bmatrix} \bar{P}_f \\ p_f^T \end{bmatrix} \in \mathbb{R}^{N+1,r}.$$

Assuming $\bar{P}_f$ and the original data P_T are both centered, we have two approaches.

Aproach 1: the Procrustes approach

Solve the following Procrustes problem

$$\begin{array}{cl} \min_Q & \|P_T - \bar{P}_f Q\|_F^2 \\ \text{s.t.} & Q^T Q = I_r. \end{array} \tag{22}$$

If $\bar{P}_f^T P_T =: U_f \Sigma_f V_f^T$, the optimal solution to (22) is $Q^* := U_f V_f^T$. The recovered position of the source is then $\boxed{p_c^T = p_f^T Q^*}$.

Approach 2: the least square approach

The second approach is to solve the least square problem

$$\begin{array}{cl} \min_Q & \|P_T - \bar{P}_f Q\|_F^2 \\ \text{s.t.} & Q \in \mathbb{R}^{r\times r}. \end{array} \tag{23}$$

The least square solution is $\bar{Q} = \bar{P}_f^{\dagger} P_T$. The recovered position of the source is then $\boxed{p_c^T = p_f^T \bar{Q}}$.

Use randomly generated data with an error proportional to the distance to each tower. The proportionality is given by η.

$$D_{n+1,i} = D_{i,n+1} = \left[\bar{d}_i \left(1 + \varepsilon_i\right)\right]^2, \tag{24}$$

where D is the generated **EDM** and $\varepsilon \in U(-\eta, \eta)$.

Error η	$\eta = 0.002$			$\eta = 0.02$			$\eta = 0.2$		
# Sensors	5	10	15	5	10	15	5	10	15
L-NEDM	0.0045	0.0014	0.0010	0.0408	0.0140	0.0120	0.3550	0.1466	0.1153
P-NEDM	0.0025	0.0013	0.0010	0.0231	0.0133	0.0117	0.2813	0.1385	0.1171
SDR	0.0024	0.0014	0.0010	0.0223	0.0137	0.0119	0.2739	0.1373	0.1164
L-FNEDM	0.0042	0.0013	0.0010	0.0356	0.0141	0.0119	0.2910	0.1395	0.1061
P-FNEDM	0.0024	0.0013	0.0010	0.0237	0.0134	0.0118	0.2623	0.1360	0.1088

Table: The mean relative error c_{re}^M of 100 simulations for varying amount of sensors and error factors with no outliers for dimension $r = 3$.

Error η	$\eta = 0.005$			$\eta = 0.05$			$\eta = 0.15$		
# Sensors	5	10	15	5	10	15	5	10	15
L-NEDM	0.0101	0.0033	0.0027	0.0970	0.0328	0.0262	0.2473	0.1037	0.0786
P-NEDM	0.0070	0.0031	0.0027	0.0610	0.0320	0.0262	0.1925	0.1041	0.0760
SDR	0.0071	0.0031	0.0027	0.0576	0.0322	0.0261	0.1933	0.1030	0.0779
L-FNEDM	0.0090	0.0032	0.0026	0.0800	0.0311	0.0255	0.2151	0.1001	0.0769
P-FNEDM	0.0069	0.0031	0.0027	0.0536	0.0310	0.0258	0.1914	0.1000	0.0772

Table: The mean relative error c_{re}^M of 100 simulations for varying amount of sensors and error factors with no outliers for dimension $r = 3$.

For each pair (n, η) and one hundred solved instances, calculate the mean of the relative error c_{re}^M for method M.

$c_{n,\eta,M} =$ mean over 100 instances, for n towers, with error factor η and method M.

Compute the *performance ratio*,

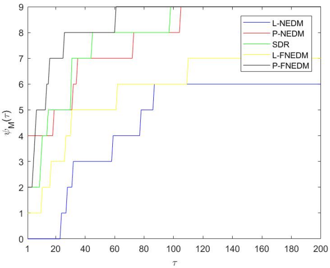
$$r_{n,\eta,M} = \frac{c_{n,\eta,M}}{\min\{c_{n,\eta,M} : M \in \mathcal{M}\}},$$

and the function,

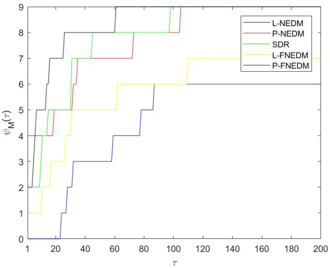
$$\psi_M(\tau) = \frac{|\{(n,\eta) : r_{n,\eta,M} \le \tau\}|}{|\mathcal{M}|}.$$

The performance profile is a plot of $\psi_M(\tau)$ for $\tau \in (1, +\infty)$ and all choices of $M \in \mathcal{M}$.

Performance Profile



(a) $\eta = (0.002, 0.02, 0.2)$



(b) $\eta = (0.0005, 0.001, 0.005, 0.01, 0.05, 0.15)$

Figure: Performance Profiles for $\psi_M(\tau)$ with $n = [5, 10, 15]$, $r = 3$, no outliers.

Thank you for your attention!

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