Homothetic packings of centrally symmetric convex bodies

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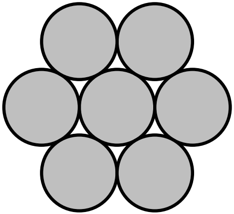
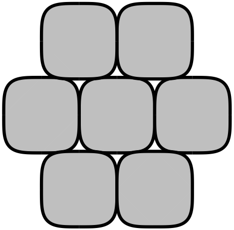
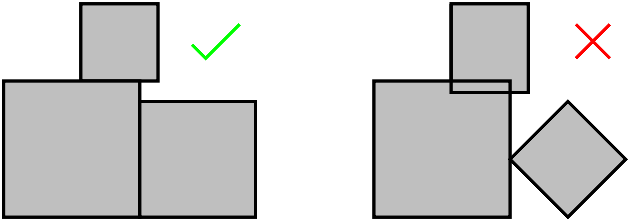
Centrally symmetric convex body packings

- **Convex body** $C \subset \mathbb{R}^d$: a compact convex set with non-empty interior.
- **Centrally symmetric (c.s.) convex body** $C \subset \mathbb{R}^d$: a convex body where $x \in C \Leftrightarrow -x \in C$.
- **Homothetic packing of** C (or C-**packing for short**): a packing (i.e. no interior overlaps) of sets of the form $rC + x$ for some $r > 0$, $x \in \mathbb{R}^d$.





More examples



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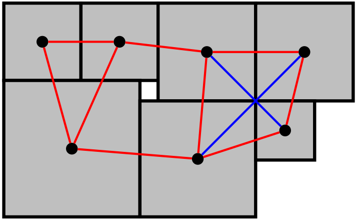

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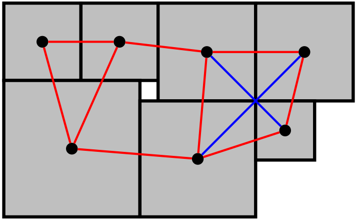

For homothetic c.s. convex bodies, the induced embedding of the contact graph is always planar if C is not a parallelogram (Danzer-Grünbaum, 1962).

Known results

Strictly convex: The boundary does not contain any line intervals with positive length.
Smooth: Every point of the boundary has at most one tangent hyperplane.

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If C is also strictly convex and G is a triangulation of a sphere, then this C-packing will be unique up to the choice of position for the bounding triangle (Schramm 1991).

Disc packings

Generic set: A finite set of real numbers that are algebraically independent over the rational numbers.

$(2, k)$-**sparse graph** ($k \in \{2, 3\}$): A graph G where for each subgraph $G' = (V', E')$ with $|E'| \geq 1$ we have $|E'| \leq 2|V'| - k$.

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Outcome: The n coins will now have at most $2n - 3$ contacts between them.

Generalising the previous result

Theorem (D– 2020+)

Let $C \subset \mathbb{R}^2$ be a strictly convex and smooth c.s. convex body. Then for almost all $r \in \mathbb{R}^{|V|}_{>0}$, any C-packing with radii r will have a planar $(2,2)$-sparse contact graph.

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What about the converse?

Conjecture

For every $(2,3)$*-sparse planar graph* $G = ([n], E)$*, there exists a* $\mathbb{D}$*-packing of* n *discs with generic radii where the contact graph is* G*.*

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Theorem (D– 2020+)

There exists a comeagre subset $\mathcal{G}$ of the c.s. convex bodies in the plane** where for each $C \in \mathcal{G}$ the following are equivalent:*

(i) *$G = (V, E)$ is a $(2,2)$-sparse planar graph.*

(ii) *There is a set $R \subset \mathbb{R}^{|V|}_{>0}$ with positive measure where for every $r \in R$ there exists $p \in \mathbb{R}^{2|V|}$ so that (G, p, r) is a C-packing.*

*A countable intersection of open dense sets. The ambient space is a Baire space if and only if every comeagre subset is also dense.

**The topology is generated by the Hausdorff distance. The set is locally compact, and hence is a Baire space by the Baire category theorem.

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Outcome: With respect to our choice of radii, there is a positive probability that you can arrange the n coins (without altering their orientations) so that their contact graph is G.

Open questions

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- Can similar methods be applied to packings where homotheticity is not required? I.e., you can rotate the coins as you slide them around.
- What can we say about packings in higher dimensions?

References

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